JOURNAL OF FUNCTIONAL ANALYSIS 14, 172–235 (1973)

Scattering Theory for Dissipative Hyperbolic Systems

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Communicated by the Editors

Received June 8, 1973

DEDICATED TO BELA SZ.-NAGY IN HIS SIXTIETH YEAR

An abstract theory of scattering is developed for dissipative hyperbolic systems, a typical example is the wave equation: $u_{tt} = \Delta u$ in an exterior domain with lossy boundary conditions: $u_n + \alpha u_t = 0$, $\alpha \geqslant 0$. In this theory, as in an earlier theory developed by the authors for conservative systems, a central role is played by two distinguished subspaces of data common to both the perturbed and unperturbed problems. Associated with each subspace is a translation representation of the unperturbed system. When these representations coincide they provide a convenient tool for extending the data so as to include a large class of generalized eigenfunctions for both the perturbed and unperturbed generators. The scattering matrix is characterized in terms of these generalized eigenfunctions; it is shown to be meromorphic in the whole complex plane and holomorphic in the lower half plane. The zeroes and poles of the scattering matrix correspond, respectively, to incoming and outgoing generalized eigenfunctions of the perturbed generator. In the second part of the paper the assumptions introduced in the abstract theory are verified for the wave equation problem cited above.

* This work was supported in part by the United States Atomic Energy Commission under Contract AT(30-1)-1480, by the National Science Foundation under Grant No. GP-8867 and by the United States Air Force under Contract AF-F44620-71-C-0037.

INTRODUCTION

Most of scattering theory to date has dealt with energy or mass conserving systems and the corresponding groups of unitary operators. However, a perfectly conservative system is an idealization which can only be approximated in nature. It is, therefore, important to extend the theory to dissipative systems. Several authors [1, 2, 5, 10, 11] have already explored this problem for quantum mechanical systems for which the perturbed system still generates what is basically a group of operators. The perturbation problem for semigroups of operators was first treated by Lin [9] in 1969. Our purpose in this paper is to present a scattering theory for dissipative hyperbolic systems based on the ideas introduced by the authors in [6] and [7] which can also handle perturbed systems which do not generate groups of operators.

In an appendix we shall present an intrinsic scattering theory for dissipative systems, i.e., one that makes no reference to an unperturbed system. Here we suppose that there is an unperturbed system, described by a one-parameter group of unitary operators $U_0(t)$, acting on a Hilbert space H_0; we denote by A_0 the generator of U_0. We suppose that there are two distinguished closed subspaces D_- and D_+, with the following properties:

$$\begin{aligned}
&\text{(i)} \quad U_0(t)\, D_- \subset D_- \quad \text{for} \quad t \leqslant 0,\\
&\phantom{\text{(i)}} \quad U_0(t)\, D_+ \subset D_+ \quad \text{for} \quad t \geqslant 0.\\
&\text{(ii)} \quad \bigwedge U_0(t) D_- = \{0\}, \qquad \bigwedge U_0(t)\, D_+ = \{0\}.\\
&\text{(iii)} \quad \overline{\bigvee U_0(t)\, D_+} = H_0, \qquad \overline{\bigvee U_0(t)\, D_-} = H_0.
\end{aligned} \tag{1}$$

According to the translation representation theorem (cf., [6, Chapter 2]) there correspond to D_+ and D_- unitary representations $\mathscr{I}_\pm$ of H_0 as vector valued functions on the real axis $\mathbb{R}$:

$$\begin{aligned}
\mathscr{I}_+ &: H_0 \to L_2(\mathbb{R}, N),\\
\mathscr{I}_- &: H_0 \to L_2(\mathbb{R}, N),
\end{aligned} \tag{2}$$

such that

$$\begin{aligned}
\mathscr{I}_+ &: D_+ \to L_2(\mathbb{R}_+, N),\\
\mathscr{I}_- &: D_- \to L_2(\mathbb{R}_-, N),
\end{aligned} \tag{3}$$

where N denotes an auxiliary Hilbert space; and

$$\mathscr{I} U_0(t) = \mathscr{T}(t)\mathscr{I}; \tag{4}$$

here $\mathscr{T}(t)$ denotes right translation by t. We $\mathscr{I}_-$ call the *incoming* and $\mathscr{I}_+$ the *outgoing representation.*

We note that Fourier transformation

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} k(s)\, e^{i\sigma s}\, ds$$

changes translation representation into spectral representation; when applied to $\mathscr{I}_+$ and $\mathscr{I}_-$ the resulting spectral representations take D_+ into the space of vector valued functions analytic in the upper half plane $\operatorname{Im} \sigma > 0$, D_- into the space of vector valued functions analytic in the lower half plane, respectively. It is the translates of D_+ and D_-, rather then D_+ and D_- themselves that enter into the theory. We shall denote these as follows:

$$D_+^a = U_0(a)\, D_+, \qquad D_-^a = U_0(-a)\, D_-, \qquad a \geqslant 0. \tag{5}$$

We turn now to the perturbed system; we suppose it to be described by a semigroup of contraction operators $T(t)$, $t \geqslant 0$, acting on a Hilbert space H. The relation between U_0, H_0, and T, H is that for some $\rho > 0$, H contains D_+^ρ and D_-^ρ as subspaces and that $U_0(t)$ and $T(t)$ coincide on D_+^ρ, whereas $U_0(-t)$ and $T^*(t)$ coincide on D_-^ρ:

$$T(t)g = U_0(t)g \qquad \text{for all} \quad g \text{ in } D_+^\rho, \tag{6}$$

$$T^*(t)f = U_0(-t)f \qquad \text{for all} \quad f \text{ in } D_-^\rho. \tag{6}^*$$

Since $T(t)$ and $T^*(t)$ agree with $U_0(t)$ and $U_0(-t)$ on D_+^ρ and D_-^ρ, respectively, it follows that they inherit some of the properties of U_0 on these subspaces. The relations (1 i) and (1 ii) are such properties, and when combined with (6) and (6)* they yield

$$\begin{aligned} &\text{(i)} \quad T^*(t)\, D_-^\rho \subset D_-^\rho, \qquad T(t)\, D_+^\rho \subset D_+^\rho, \\ &\text{(ii)} \quad \bigwedge T^*(t)\, D_-^\rho = \{0\}, \qquad \bigwedge T(t)\, D_+^\rho = \{0\}. \end{aligned} \tag{7}$$

Property (1 iii) refers to actions of U_0 which carry $D_\pm^\rho$ outside of themselves; therefore, (1 iii) implies nothing about the action of T and T^*. Instead, we have to assume an analogous property for T and T^*; the verification of this property in the application presented in Part II is in fact the most intricate technical part of our paper.

To state the needed property we introduce two projections P_+^ρ and P_-^ρ, defined as orthogonal projections onto the orthogonal complements of D_+^ρ and D_-^ρ, respectively. That is, P_+^ρ removes the D_+^ρ component, P_-^ρ removes the D_-^ρ component. The analog of (1 iii) is then: For every f in H

$$\begin{aligned} \lim_{t\to\infty} P_-^\rho T^*(t) f &= 0, \\ \lim_{t\to\infty} P_+^\rho T(t) f &= 0. \end{aligned} \tag{8}$$

Property (8) expresses the fact that under the action of T or T^*, data eventually end up in D_+^ρ, respectively D_-^ρ.

The following two examples will illustrate the kind of systems we are concerned with

EXAMPLE 1. A very transparent model for our theory is: $H_0 = L_2(R)$, $[U_0(t)f](s) = f(s-t)$, $D_-^\rho = L_2((-\infty, -\rho))$, $D_+^\rho = L_2((\rho, \infty))$; $H = L_2(R)$ and $[T(t)f](s) = f(s-t)$ if $s \leqslant 0$ or $s > t$ and $= 0$ otherwise.

EXAMPLE 2. A more interesting example is provided by the acoustic equation. Here the unperturbed system satisfies the equation

$$u_{tt} = \Delta u \qquad \text{in } \mathbb{R}^n, \tag{9}$$

the Hilbert space H_0 is the set of all initial data $d = (f_1, f_2)$:

$$u(x, 0) = f_1(x), \qquad u_t(x, 0) = f_2(x)$$

with energy norm:

$$\| d \|_E^2 = \int [| \partial_x f_1 |^2 + | f_2 |^2] \, dx. \tag{10}$$

The subspace D_-^ρ $[D_+^\rho]$ consists of the set of all initial data for solutions which vanish in the truncated backward [forward] cone: $| x | \leqslant \rho - t$, $t \leqslant 0$ $[| x | \leqslant \rho + t, t \geqslant 0]$; here ρ is chosen so that the scattering obstacle is contained in the ball $\{| x | < \rho\}$. The perturbed system satisfies the acoustic equation (9) in the exterior G of the obstacle and dissipative boundary conditions

$$u_n + \alpha u_t = 0 \qquad \text{on} \quad \partial G, \qquad \alpha \geqslant 0; \tag{11}$$

here n denotes the outward normal to G on ∂G. As we show in Section 9 in this example property (8) is a consequence of local energy decay.

Part I of this paper is devoted to the abstract theory; it is organized as follows: In Section 1 we define the scattering operator S as

$$S = \operatorname*{st\,lim}_{t\to\infty} U_0(-t)\, J^*T(2t)\, JU_0(-t), \tag{12}$$

where J maps H_0 into H. We show that if conditions (1 i)–(1 iii), (6), (7 i)–(7 ii), and (8) hold, the limit above exists. The operator S commutes with U_0, from which it follows that in a translation representation for U_0 the operator S acts as convolution with an operator valued distribution which we denote as $S(s)$

$$\mathscr{I}S = S * \mathscr{I}. \tag{13}$$

In the corresponding spectral representation S acts as multiplication by the Fourier transform $\mathscr{S}$ of S;

$$\mathscr{S}(\sigma) = \int S(s)\, e^{is\sigma}\, ds. \tag{14}$$

$\mathscr{S}(\sigma)$ is called the *scattering matrix*.

In many important applications the two translation representations $\mathscr{I}_-$ and $\mathscr{I}_+$ discussed earlier coincide; when this occurs one can show that in this representation S is *causal* in the sense that it maps $L_2(\mathbb{R}_- - \rho, N)$ into $L_2(\mathbb{R}_- + \rho, N)$. Consequently, in this case the function $S(s)$ has its support in $\mathbb{R}_- + 2\rho$, and its Fourier transform, the scattering matrix $\mathscr{S}(\sigma)$, is then analytic in the lower half plane $\operatorname{Im}\sigma < 0$ where it is of exponential growth. Even when S is not causal, it is often possible to continue $\mathscr{S}(\sigma)$ into the lower half plane from $\mathbb{R}_+$ or $\mathbb{R}_-$, but usually not from both simultaneously; a general result of this kind is given at the end of Section 1.

If we convolve S with an exponential function $e = e^{-izs}n$, n in N, we get, using (14),

$$\begin{aligned} S * (e^{-izs}n) &= \int S(r)\, e^{-iz(s-r)} n\, dr = e^{-izs} \int S(r)\, e^{izr} n\, dr \\ &= e^{-izs}\mathscr{S}(z)n. \end{aligned} \tag{15}$$

This suggests that the scattering matrix $\mathscr{S}(z)$ can be studied by observing the action of the scattering operator S on data whose translation representation is exponential. Since exponential functions are not in L_2 over $\mathbb{R}$, it follows that they do not represent elements in H_0. In order to be able to talk of such elements, we extend the spaces H_0 and H in Section 2 and then define suitable extensions for the operators U_0, T, $P_\pm^a$, A_0, and A to the extended spaces. In Section 3 we use this modest machinery to legitimize formula (15).

In Section 4 we introduce the associated semigroup

$$Z(t) = P_+^\rho T(t) P_-^\rho \qquad \text{on} \quad K = H \ominus (D_-^\rho \oplus D_+^\rho). \tag{16}$$

This turns out to be a useful tool in the study of the scattering matrix. In fact if the resolvent of the generator B of Z is meromorphic then the set of poles of $\mathscr{S}(z)$ coincide with $-i$ times the spectrum of B. In Section 5 we show that the poles of $\mathscr{S}(z)$ correspond to outgoing (orthogonal to D_-^ρ) generalized eigenfunctions of $A : Aa = iz\, a$ and the zeros of $\mathscr{S}(z)$ to the incoming (orthogonal to D_+^ρ) generalized eigenfunctions of A. The zeros of $\mathscr{S}(z)$ in the upper half plane correspond to ordinary eigenfunctions of A in H.

The spectrum of A always contains a continuous part filling out the imaginary axis. In Section 6 we present a criterion for the rest of the spectrum of A to be point spectra and hence determined by the zeros of $\mathscr{S}(z)$

Part II of this paper applies the abstract theory to Example 2 in the introduction, that is the wave equation in the exterior of an obstacle, subject to a dissipative boundary condition. In Section 7 we show that the solutions of the wave equation in free space, with the classical energy norm, and with D_+^0 and D_-^0 defined as in Example 2, satisfy properties (1 i)–(1 iii); we find explicitly the corresponding translation representations $\mathscr{I}_-$ and $\mathscr{I}_+$ in terms of the Radon transform. When the number of space dimensions is odd, $\mathscr{I}_- = \mathscr{I}_+$; for n even $\mathscr{I}_-$ is related to $\mathscr{I}_+$ via the Hilbert transform.

In Section 8 we show that the solutions in the exterior domain form a dissipative semigroup, and study the spectrum of the infinitesimal generator A. In Section 9 we show that for every solution with finite energy the energy contained in any compact region tends to 0 as t tends to ∞; from this we deduce that property (8) is satisfied. In Section 10 we connect the abstract description of the scattering matrix given in Section 3 to the usual description in terms of reflection coefficients for an odd number of spatial dimensions. Finally, we show that A satisfies the criterion of Section 6 and, hence, has only a point spectrum in addition to a continuous part which fills out the imaginary axis.

I. AN ABSTRACT FRAMEWORK

1. The Wave and Scattering Operators

Throughout this part we assume that

(a) $U_0(t)$ is a group of unitary operators satisfying conditions (1 i)–(1 iii);

(b) $T(t)$ is a semigroup of contractions satisfying conditions (7 i)–(7 ii) and (8);

(c) U_0 and T are related through conditions (6) and (6)* which can be restated as

$$U_0(-t)\, T(t) = I \qquad \text{on} \quad D_+^\rho, \tag{1.1}$$

$$U_0(t)\, T^*(t) = I \qquad \text{on} \quad D_-^\rho. \tag*{(1.1)*}$$

We start by deducing from (1.1) and (1.1)* a further useful relation between U_0 and T

$$T(t)\, U_0(-t) = I \qquad \text{on} \quad D_-^\rho, \tag{1.2}$$

$$T^*(t)\, U_0(t) = I \qquad \text{on} \quad D_+^\rho. \tag*{(1.2)*}$$

The proof relies on the following general proposition about Hilbert space operators.

Lemma 1.1. *Let W be an operator mapping one Hilbert space H_0 into another H, satisfying the following conditions*:

(a) *W is contractive, i.e., $\| W \| \leqslant 1$;*

(b) *W is the identity on some subspace D common to both H_0 and H.*

Then

(c) *W maps the orthogonal complement of D in H_0 into the orthogonal complement of D in H;*

(d) *W^* is the identity on D.*

Proof. Let x be any element of D, y any element of $D^\perp$ in H_0. By property (b),

$$W(x + \epsilon y) = x + \epsilon Wy.$$

By property (a)

$$\| x + \epsilon Wy \|^2 \leqslant \| x + \epsilon y \|^2.$$

Since y is orthogonal to x, the coefficient of ϵ on the right in the inequality above is zero. Therefore, it must be zero also on the left, which proves part (c). This shows that D and $D^\perp$ completely reduce W, from which part (d) follows.

We should like to apply Lemma 1.1 to $W = U_0(-t)\,T(t)$. However, this operator is not well defined because U_0 and T act on different spaces which have in common only D_-^ρ and D_+^ρ (not necessarily orthogonal). We, therefore, need linear mappings that take us from H_0 to H and back. Let J_0 and J be such maps

$$J_0 \colon H_0 \to H,$$
$$J \colon H \to H_0 .$$

We only require these maps to be linear and continuous and to act like the identity on the common part of H_0 and H, which is $D_+^\rho + D_-^\rho$. For instance, we could choose J_0 and J to be orthogonal projections of their domain spaces onto $\overline{D_+^\rho + D_-^\rho}$. Note that for this choice J is the adjoint of J_0; such a choice is convenient for some calculations, and we shall use it whenever we need it.

We can now apply Lemma 1.1 to $W = U_0(-t)\,JT(t)$ and $D = D_+^\rho$. Clearly, $\| W \| \leqslant 1$ and according to (1.1) condition (b) is fulfilled. Hence, conclusion (d) follows so that $W^* = T^*(t)\,J_0 U_0(t)$ acts like the identity on D_+^ρ. This proves (1.2)* and (1.2) follows similarly from (1.1)*.

In the important special case when D_-^ρ and D_+^ρ are orthogonal, we can decompose H into three orthogonal components

$$H = D_-^\rho \oplus K^\rho \oplus D_+^\rho,$$

where $K^\rho = H \ominus (D_-^\rho \oplus D_+^\rho)$. It is instructive to observe the action of $T(t)$ on these three components. From properties (1.1) and (1.2) and the orthogonality of the three components one can easily deduce

(i) $T(t)$ maps D_-^ρ into all three components, but the $D_+^{\rho+t}$ component of D_+^ρ is uneffected;

(ii) $T(t)$ maps K^ρ into K^ρ and D_+^ρ, and again the $D_+^{\rho+t}$ component of D_+^ρ is uneffected;

(iii) $T(t)$ maps D_+^ρ into itself.

It is useful to think of D_-^ρ, K^ρ, and D_+^ρ as past, present and future and to measure time in the past and future by the parameter s in the incoming and outgoing translation representations. If we think of the action of $T(t)$ as the flow of time, then the above observations on the action of $T(t)$ can be described picturesquely as follows: *The past influences the present and the future, and the present influences the present and the future. No instant of time t_0 can have any influence on a later time t_1 in less time than $t_1 - t_0$.*

Such action can be described as *causal*, with signal speed $\leqslant 1$.

Next we turn to studying the asymptotic relation between the actions of U_0 and T.

THEOREM 1.2. *Define the operators $W_1(t)$ and $W_2(t)$ by*

$$W_1(t) = T(t)\, J_0 U_0(-t), \tag{1.3$_1$}$$

$$W_2(t) = U_0(-t)\, JT(t), \tag{1.3$_2$}$$

then the limits

$$W_1 = \underset{t\to\infty}{\text{st. lim}}\, W_1(t), \tag{1.4$_1$}$$

and

$$W_2 = \underset{t\to\infty}{\text{st. lim}}\, W_2(t), \tag{1.4$_2$}$$

exist for any admissible J_0 and J, and are independent of the choices of J_0 and J. W_1 and W_2 are called the wave operators.

Proof. Let f be of the form

$$f = U_0(s)k, \qquad k \text{ in } D_-^{\rho}, \tag{1.5}$$

then

$$W_1(t)f = T(t)\, J_0 U_0(-t)f = T(t)\, J_0 U_0(s-t)k. \tag{1.6}$$

It follows from (1.2) (with $t - s$ in place of t) that for $t > s$, (1.6) equals $T(s)k$. Being independent of t for $t > s$, the limit of (1.6) as t tends to ∞ exists for every f of form (1.5). According to condition (1 iii) elements of the form (1.5) are dense in H_0; since the operators $W_1(t)$ are uniformly bounded for all t, convergence on a dense set implies convergence everywhere. It is clear from the above that this limit does not depend on the choice of J_0.

To prove the existence of W_2, let f be any element of H, and decompose $T(t)f$ as a sum of elements of D_+^{ρ} and its orthogonal complement

$$T(t)f = d(t) + e(t), \tag{1.7}$$

where

$$d(t) \in D_+^{\rho} \qquad \text{and} \qquad e(t) \perp D_+^{\rho}.$$

Using (1.7) and the fact that d is in D_+^ρ and that J is the identity on D_+^ρ, we get

$$W_2(t)f = U_0(-t)\,JT(t)f = U_0(-t)\,d(t) + U_0(-t)\,Je(t). \tag{1.8}$$

Similarly, for $s > 0$,

$$\begin{aligned} W_2(t+s)f &= U_0(-t-s)\,JT(t+s)f \\ &= U_0(-t-s)\,T(s)\,d(t) + U_0(-t-s)\,JT(s)\,e(t). \end{aligned}$$

According to (1.1), T and U_0 agree on D_+^ρ, so we get

$$W_2(t+s)f = U_0(-t)\,d(t) + U_0(-t-s)\,JT(s)\,e(t). \tag{1.9}$$

Subtract (1.8) from (1.9)

$$W_2(t+s)f - W_2(t)f = U_0(-t-s)\,JT(s)\,e(t) - U_0(-t)\,Je(t). \tag{1.10}$$

By assumption (8), $\| e(t)\|$ tends to zero as t tends to ∞; this shows that the right side of (1.10) tends to 0, which proves the convergence of (1.10) to a limit. The independence of this limit from the choice of J follows from formula (1.8).

We now list some properties of the wave operators which will be used subsequently.

THEOREM 1.3. (a) *U_0 and T are intertwined by both W_1 and W_2, i.e.,*

$$W_1 U_0(s) = T(s)\,W_1, \tag{1.11}$$

and

$$W_2 T(s) = U_0(s)\,W_2; \tag{1.12}$$

(b)

$$W_1 = I \quad \text{on} \quad D_-^\rho, \tag{1.13}$$

and

$$W_2 = I \quad \text{on} \quad D_+^\rho; \tag{1.14}$$

(c) *For any $a \geqslant \rho$, W_1 maps $(D_-^a)^\perp$ in H_0 into $(D_-^a)^\perp$ in H and W_2 maps $(D_+^a)^\perp$ in H into $(D_+^a)^\perp$ in H_0;*

(d) *Suppose D_+^ρ and D_-^ρ are orthogonal; then for any $a \geqslant 0$, W_1 maps $U_0(a)\,D_-^\rho$ into the orthogonal complement of $D_+^{a+\rho}$, W_2 maps $D_-^{a+\rho}$ into the orthogonal complement of $U_0(-a)\,D_+^\rho$.*

Proof. From the definition (1.3) of $W_1(t)$, $W_2(t)$, we see that

$$W_1(t)\, U_0(s) = T(s)\, W_1(t - s),$$

and

$$W_2(t)\, T(s) = U_0(s)\, W_2(t + s).$$

Hence, letting t tend to ∞ we deduce the intertwining relations (1.11) and (1.12).

Again from the definition of $W_1(t)$, $W_2(t)$ and the relations (1.1) and (1.2) we see that

$$W_2(t) = I \qquad \text{on} \quad D_+^\rho,$$

$$W_1(t) = I \qquad \text{on} \quad D_-^\rho.$$

Letting t tend to ∞ we deduce (1.13) and (1.14).

Relations (c) follow from part (c) of Lemma 1.1 on using part (b) of Theorem 1.3.

To prove the first part of (d) we write $D_+^{a+\rho}$ as $U_0(a)\, D_+^\rho$ and state the result to be proved in the following slightly symbolic form:

$$(W_1 U_0(a)\, D_-^\rho,\ U_0(a)\, D_+^\rho) = 0.$$

Using the intertwining property (1.11) we can replace $W_1 U_0(a)$ by $T(a)\, W_1$. Since by (1.13), $W_1 = I$ on D_-^ρ, we can rewrite the foregoing as

$$T(a)\, D_-^\rho,\ U_0(a)\, D_+^\rho) = 0,$$

which is the same as

$$(D_-^\rho,\ T^*(a)\, U(a)\, D_+^\rho) = 0.$$

By (1.2)*, $T^*(a)\, U_0(a) = I$ on D_+^ρ, so the above relation merely expresses the orthogonality of D_-^ρ and D_+^ρ; but that was assumed to be true, so the proof of the first part of (d) is complete.

The second part of (d) is handled similarly, $D_-^{a+\rho}$ is written as $U_0(-a)\, D_-^\rho$, and the result to be proved is written as

$$(W_2 U_0(-a)\, D_-^\rho,\ U_0(-a)\, D_+^\rho) = 0.$$

Moving $U_0(-a)$ across the inner product and using the second intertwining property (1.12) we get

$$(W_2 T(a)\, U_0(-a)\, D_-^\rho,\ D_+^\rho) = 0.$$

By (1.2), $T(a)\, U_0(-a) = I$ on $D_-{}^\rho$, so after shifting W_2 we can rewrite the above as

$$(D_-{}^\rho, W_2{}^* D_+{}^\rho) = 0.$$

It follows from (1.14) upon the application of part (d) of Lemma 1.1 that $W_2{}^* = I$ on $D_+{}^\rho$; so the above relation expresses the orthogonality of $D_+{}^\rho$ and $D_-{}^\rho$. This completes the proof of Theorem 1.3.

We turn now to the *scattering operator*, defined by

$$S = W_2 W_1 . \tag{1.15}$$

Some of its properties are summarized in the following theorem.

THEOREM 1.4. (a) *S commutes with U_0:*

$$U_0(s) S = S U_0(s); \tag{1.16}$$

(b) *$S - I$ maps $D_-{}^\rho$ into the orthogonal complement of $D_+{}^\rho$;*

(c) *If $D_-{}^\rho$ and $D_+{}^\rho$ are orthogonal, S maps $D_-{}^\rho$ into the orthogonal complement $D_+{}^\rho$.*

Proof. Part (a) for $s \geqslant 0$ is an immediate consequence of the intertwining relations (a) of Theorem 1.3 and the definition (1.15) of S. On multiplying both sides of (1.16) by $U_0(-s)$ we see that this relation holds for all real s. To prove part (b), take f in $D_-{}^\rho$; then according to (1.13) of Theorem 1.3

$$W_1 f = f.$$

We conclude, using the definition (1.15) of S that

$$Sf = W_2 f.$$

Next decompose f into orthogonal parts with respect to $D_+{}^\rho$

$$f = d + e, \qquad d \text{ in } D_+{}^\rho, \qquad e \perp D_+{}^\rho, \tag{1.17}$$

then

$$Sf = W_2 d + W_2 e,$$

and subtracting (1.17) from this given

$$Sf - f = W_2 d - d + W_2 e - e.$$

According to parts (b) and (c) of Theorem 1.3, $W_2 d = d$ and $W_2 e$ is orthogonal to D_+^ρ; this shows that

$$Sf - f = W_2 e - e,$$

is orthogonal to D_+^ρ, as asserted in part (b). Part (c) is a corollary of part (b).

Since S commutes with U_0, it follows that in a spectral representation for U_0 the operator S acts as multiplication by some function $\mathscr{S}(\sigma)$. This is true even if we employ two different spectral representations of H_0 as the domain and as the range of S. We shall in fact take as the domain of S the *incoming* and as the range the *outgoing spectral representations* of U_0, defined in the introduction, and define the *scattering matrix* to be the multiplying factor $\mathscr{S}(\sigma)$ describing the action of S in these particular representations. Since these representations employ vectorvalued functions, $\mathscr{S}(\sigma)$ is an operator valued function.

Since the identity also commutes with U_0, it follows that when the above representations are employed for domain and range, the identity acts as multiplication by some factor which we denote by $\mathscr{K}(\sigma)$.

We noted in the introduction that in the incoming spectral representation D_-^ρ is represented by analytic functions in the lower half plane $\operatorname{Im} \sigma < 0$, bounded by $e^{\rho \operatorname{Im} \sigma}$, and in the outgoing spectral representation the orthogonal complement of D_+^ρ is represented by functions analytic in the lower half plane, bounded by $e^{-\rho \operatorname{Im} \sigma}$. According to part (b) of Theorem 1.4, $S - I$ maps D_-^ρ into the complement of D_+^ρ; consequently, we conclude from the foregoing that multiplication by $\mathscr{S}(\sigma) - \mathscr{K}(\sigma)$ maps analytic functions in the lower half plane, bounded by $e^{\rho \operatorname{Im} \sigma}$, into analytic function in the lower half plane, bounded by $e^{-\rho \operatorname{Im} \sigma}$. It follows, see [6], that $\mathscr{S}(\sigma) - \mathscr{K}(\sigma)$ is *itself analytic in the lower half plane, and bounded by* $e^{-2\rho \operatorname{Im} \sigma}$.

Suppose the incoming and outgoing representations coincide; then $\mathscr{K}(\sigma) \equiv 1$ and we conclude that $\mathscr{S}(\sigma)$ itself can be continued analytically into the lower half plane where it is of exponential growth. In Section 3 we shall rederive this result.

In one of the applications studied in Part II, the factor $\mathscr{K}(\sigma)$ is $\not\equiv 1$ but can be continued analytically into the lower half plane from either the positive or the negative real axis. It then follows from the analyticity of $\mathscr{S} - \mathscr{K}$ that the same is true for $\mathscr{S}(\sigma)$.

2. Extensions of the Spaces H_0 and H

Throughout the remainder of Part I we assume that *the incoming and outgoing translation representations for U_0 described in the intro-*

duction are the same. We shall denote this representation by the unsubscripted symbol $\mathscr{I}$. Note that D_-^{ρ} and D_+^{ρ} are automatically orthogonal in this case.

We define the space $\overline{H_0}$ *to consist of all N-valued locally* L_2 *functions on* $\mathbb{R}$. $\overline{H_0}$ is an extension of $L_2(\mathbb{R}, N)$; identifying $L_2(\mathbb{R}, N)$ with H_0 via the representation $\mathscr{I}$ makes $\overline{H_0}$ an extension of H_0.

We show now how to extend the operators U_0, A_0 and $P_{\pm}{}^a$ to $\overline{H_0}$: We define $U_0(t)$ as right translation. We define A_0 to be $-\partial/\partial s$, its domain consisting of all functions whose distribution derivatives are locally L_2.

We define $P_+{}^a$ and $P_-{}^a$ on $\overline{H_0}$ as multiplication by the characteristic function of $\mathbb{R}_- + a$ and $\mathbb{R}_+ - a$, respectively.

We shall mainly use the one-sided extension of H_0 defined as consisting of all N-valued functions which are L_2 over any interval of the form (a, ∞) with a arbitrary. We denote this extension by $\overline{H_0}'$. We shall say that f_n converges to f in $\overline{H}_0'$ if $f_n - f$ in $L_2((a, \infty), N)$ for each a. Clearly U_0, A_0, and $P_{\pm}$ can also be extended as mappings of $\overline{H}_0'$ into itself.

We define similarly an extension of the space H; first we decompose H into three orthogonal parts:

$$H = D_-{}^a \oplus K^a \oplus D_+{}^a, \tag{2.1}$$

where $a > \rho$ and K^a is the orthogonal complement in H of $D_+{}^a \oplus D_-{}^a$. We then define $\overline{D_+{}^a}$ and $\overline{D_-{}^a}$ to consist of all N-valued locally L_2 functions on $\mathbb{R}_+ + a$ and $\mathbb{R}_- - a$, respectively. Identifying $D_+{}^a$ with $L_2(\mathbb{R}_+ + a, N)$ and $D_-{}^a$ with $L_2(\mathbb{R}_- - a, N)$ via the representation $\mathscr{I}$ makes $\overline{D_+{}^a}$ and $\overline{D_-{}^a}$ an extension of $D_+{}^a$ and $D_-{}^a$, respectively. We define $\overline{H}$ as

$$\overline{H} = \overline{D_-{}^a} \oplus K^a \oplus \overline{D_+{}^a}, \tag{2.2}$$

and we define the one-sided extension $\overline{H}'$ as

$$\overline{H}' = \overline{D_-{}^a} \oplus K^a \oplus D_+{}^a.$$

Note that $\overline{H}$ and $\overline{H}'$ do not actually depend on the value of a used in their definition.

We define $P_+{}^a$ as annihilating the $\overline{D_+{}^a}$ component and being the identity on the K^a and $\overline{D_-{}^a}$ components; $P_-{}^a$ is defined analogously.

On $\overline{D_+{}^a}$ we define $T(t)$ as right translation. On $\overline{D_-{}^a}$ we define $T(t)$ as right translation, provided that $a \geqslant \rho + t$; since the choice of a is arbitrary this can always be achieved. The action of $T(t)$ on K^a is the same as before carrying K^a into $\oplus (D_+{}^a \ominus D_+^{a+t})$. $T^*(t)$ is defined similarly.

To define Af for f in $\bar{H}$ we first decompose f as in (2.2)

$$f = f_- + k + f_+ . \tag{2.3}$$

Then we choose two smooth cut-off functions ζ_- and ζ_+, defined on $(-\infty, -a)$ and (a, ∞), respectively, having the property that $\zeta_\pm$ is equal to 0 near $+a$, equal to 1 near $+\infty$ and smooth in between. We write f as

$$f = \zeta_- f_- + [(1 - \zeta_-) f_- + k + (1 - \zeta_+) f_+] + \zeta_+ f_+ . \tag{2.4}$$

Note that the element in the square brackets belongs to H, whereas the other two terms lie in $\overline{D_-{}^a}$ and $\overline{D_+{}^a}$ where the actions of A and A_0 coincide. We say that f belongs to the domain of A if both f_- and f_+ have locally L_2 derivatives, and if the element in the square brackets in (2.3) belongs to the domain of A in H. Af is defined using the decomposition (2.3), with A acting as $-\partial/\partial s$ on the two outer components. Clearly, this definition does not depend on the choice of the cut-off functions and extends the action of A on H.

We turn now to the operator W_1. According to relation (1.13) of Theorem 1.3, W_1 is the identity on $D_-{}^a$; we extend W_1 by taking it to be the identity on $\overline{D_-{}^a}$. This defines a one-sided extension of W_1 as a mapping from $\overline{H_0{}'}$ to $\overline{H'}$; it is clear that W_1 is a continuous transformation.

It turns out that the operator W_2 has a two-sided extension, although we shall only use its one-sided extension. First of all, according to (1.14) W_2 is the identity on $D_+{}^a$ and can be extended as the identity to $\overline{D_+{}^a}$. To extend W_2 on the negative side we make use of part (d) of Theorem 1.3, according to which W_2 maps $D_-^{b+\rho}$ into $[U_0(-b)\, D_+{}^\rho]^\perp$. In our situation, where the incoming and outgoing representations are the same, $D_-^{b+\rho}$ is represented by functions supported on $[-\infty, -b-\rho]$, and $U_0(-b) D_+{}^\rho$ by functions supported on $[-b+\rho, \infty]$. The orthogonal complement of $U_0(-b)\, D_+{}^\rho$ in H_0 is, therefore, represented by functions supported on $[-\infty, -b+\rho]$. This analysis shows that in the present situation property (d) of W_2 amounts to a kind of causality:

For $b \geqslant 2\rho$, $\quad W_2$ *maps* $D_-^{b+\rho}$ *into* $D_-^{b-\rho}$.

We can now extend W_2 to any f in $D_-{}^a$; decompose f as a sum of functions,

$$f = \sum f_j$$

where f_j is L_2 and supported in $[-j-1, -j+1]$. Set

$$W_2 f = \sum W_2 f_j .$$

According to the foregoing the support of $W_2 f_j$ is confined to $s \leqslant -j + 1 + 2\rho$; so the above sum defining $W_2 f$ is locally finite and defines a locally L_2 function. For the same reason the value of $W_2 f$ does not depend on the particular decomposition used for f. In this fashion we can define W_2 to map $\overline{H'}$ into $\overline{H_0'}$ and it is clear from this construction that W_2 is continuous.

The scattering operator S is the product $W_2 W_1$; it can, therefore, be extended as an operator mapping $\overline{H_0'}$ into $\overline{H_0'}$ continuously.

It is easy to show that these extensions are independent of the value of a, and that the extended operators retain most of their previous properties. In particular *the intertwining properties* (1.11) *and* (1.12) *hold for the extended operators* W_1, W_2, U_0, *and* T. It follows that S *extended commutes with* U_0 *extended* and that S is *causal*:

The values of f at $s \leqslant a$ do not influence the values of Sf at $s > a + 2\rho$.

Remark. A less direct but more systematic method of obtaining these extensions is to regard $\overline{H}_0[\overline{H}]$ as the dual of the subspace $\underline{H}_0[\underline{H}]$, consisting of the elements in $H_0[H]$ whose incoming and outgoing components have compact support. The relevant operators can then be defined as the duals of their formal adjoints on $\underline{H}_0$ or $\underline{H}$.

3. The Scattering Matrix

We have shown in Section 2 that the scattering operator S can be extended as an operator from $\overline{H_0'}$ into $\overline{H_0'}$. Consider now the element e_0 of $\overline{H_0'}$ defined by

$$\mathscr{J} e_0 = e^{-izs} n, \qquad \operatorname{Im} z < 0, \tag{3.1}$$

where n is some element of N. Then e_0 belongs to $\overline{H_0'}$, and, hence, $e_1 = S e_0$ is defined.

Being represented by an exponential function, e_0 is an eigenfunction of the group U_0

$$U_0(t)\, e_0 = e^{izt} e_0 . \tag{3.2}$$

Since S commutes with U_0, it follows that $e_1 = S e_0$ also satisfies

$$U_0(t)\, e_1 = e^{izt} e_1 ,$$

from which it follows that e_1 is also represented by an exponential function

$$\mathscr{I} S e_0 = \mathscr{I} e_1 = e^{-izs} m. \tag{3.3}$$

Here m is an element of N, determined by z and n; clearly m depends linearly on n and it is consistent with (15) to write

$$m = \mathscr{S}(z) n. \tag{3.3$'$}$$

$\mathscr{S}(z)$ is ealled tne scattering matrix; we summarize its properties in the following theorem.

THEOREM 3.1. (a) *For any z in $\operatorname{Im} z < 0$, $\mathscr{S}(z)$ is a bounded operator on N*

$$\| \mathscr{S}(z) \| \leqslant e^{2\rho |\operatorname{Im} z|}. \tag{3.4}$$

(b) *$\mathscr{S}(z)$ depends analytically on z in the lower half plane.*

(c) *For any element f in H_0 which is represented by a function supported on $\mathbb{R}_-$,*

$$Sf = g \tag{3.5}$$

is represented by a function supported on $\mathbb{R}_- + 2\rho$. Therefore, both f and g have spectral representations which can be extended to be analytic in the lower half plane. The relation between these extended spectral representations is

$$\mathscr{S}(z) \tilde{f}(z) = \tilde{g}(z). \tag{3.6}$$

Proof. Denote as usual by P_-^0 and $P_-^{2\rho}$ the operators which remove the D_-^0 and $D_-^{2\rho}$ components, respectively. By causality, stated near the end of Section 2, it follows that $\mathscr{I} S P_-^{2\rho} e_0$ coincides with $\mathscr{I} S e_0 = \mathscr{I} e_1$ on $\mathbb{R}_+$. It follows, therefore, that

$$\| P_-^0 e_1 \| \leqslant \| S P_-^{2\rho} e_0 \|.$$

Since S is a contradiction, we deduce that

$$\| P_-^0 e_1 \| \leqslant \| P_-^{2\rho} e_0 \|.$$

These norms are easily computed explicitely

$$\| P_-^0 e_1 \| = \| m \| / (2 \mid \operatorname{Im} z \mid)^{1/2},$$

$$\| P_-^{2\rho} e_0 \| = \| n \| / (2 \mid \operatorname{Im} z \mid)^{1/2} e^{|\operatorname{Im} z| 2\rho};$$

and inequality (3.4) follows.

To prove the analyticity of $\mathscr{S}(z)$, define h in H_0 to be

$$h = p(s)k,$$

where $p(s)$ is any continuous real valued function with compact support contained in $\mathbb{R}_+$ and k is an element of N. Arguing as before we have

$$(SP_-^{2\rho}e_0, h) = (e_1, h) = (\mathscr{S}(z)n, k)\,\tilde{p}(-z);$$

here $\tilde{p}$ denotes the Fourier transform of p. The left side can be rewritten as

$$(P_-^{2\rho}e_0, S^*h),$$

this clearly depends analytically on z. But then so does the right side; dividing by $\tilde{p}(-z)$ we conclude that $\mathscr{S}(z)$ is weakly and, hence, strongly analytic, as expected.

To prove part (c) we make use of the fact that S and U_0 commute

$$SU_0(r)f = U_0(r)g.$$

It is convenient to work with the translation representation of U_0 for which $[U_0(r)g](s) = g(s-r)$. Multiply both sides by e^{-izr} and integrate with respect to r from $-R$ to R; the right side is

$$\int_{-R}^{R} g(s-r)\,e^{-izr}\,dr = e^{-izs}\int_{s-R}^{s+R} g(t)\,e^{izt}\,dt.$$

The limit of this as R tends to ∞ is

$$e^{-izs}\,\tilde{g}(z). \tag{3.7}$$

On the left side we bring the factor e^{-izr} under S and perform the integration on the argument of S; the resulting argument of S is

$$e^{-izs}\int_{s-R}^{s+R} f(t)\,e^{izt}\,dt.$$

As R tends to ∞, this tends to

$$e^{-izs}\tilde{f}(z).$$

Furthermore, the convergence holds in the L_2 sense over any s-interval

$[a, \infty]$. It follows, therefore, from the fact that S is continuous in $\overline{H_0'}$ that the left side tends to

$$S(e^{-izs}\tilde{f}(z)) \tag{3.8}$$

in the L_2 sense over any s-interval $[a, \infty]$. Using the definition (3.3), (3.3)′ of $\mathscr{S}$, with $n = \tilde{f}(z)$, we see that (3.8) can be written as

$$e^{-izs}\mathscr{S}(z)\tilde{f}(z). \tag{3.9}$$

Finally, the expressions (3.7) and (3.9) yield the desired relation (3.6). This completes the proof of Theorem 3.1.

Remark. We shall show in Section 4 that $\mathscr{S}(z)$ can be continued analytically to be meromorphic in the entire plane, and to be holomorphic on the real axis. Continuing (3.6) to the real axis gives

$$\mathscr{S}(\sigma)\tilde{f}(\sigma) = \tilde{g}(\sigma) \qquad \sigma \in \mathbb{R} \tag{3.6$'$}$$

for all f in D_-^a; however, because of (1 iii) and (1.16) we see that (3.6)′ provides the spectral representation of S for all f in H_0 and that $\mathscr{S}(\sigma)$ is in fact what is usually called the scattering matrix.

We return now to the eigenfunction e_0 of U_0 defined by (3.1). For $\operatorname{Im} z < 0$, e_0 belongs to H_0' and this lies in the domain of W_1 extended. We define e by

$$e = W_1 e_0 . \tag{3.10}$$

THEOREM 3.2. *Let e denote the element of $\overline{H}'$ defined by* (3.1), (3.10). *Then*

(a) *e is an eigenvector of T*:

$$T(t)e = e^{izt}e. \tag{3.11}$$

(b) *The $\overline{D_-^\rho}$ component of e is of the form*

$$\mathscr{I}e_- = e^{-izs}n, \tag{3.12}$$

whereas the D_+^ρ component of e is of the form

$$\mathscr{I}e_+ = e^{-izs}\mathscr{S}(z)n. \tag{3.13}$$

Proof. Apply W_1 to the eigenvalue relation (3.2) satisfied by e_0; using the first intertwining relation (1.11) we get (3.11); this completes the proof of part (a).

To prove part (b) we use the fact that $T(t)$ acts as translation on the

$D_+{}^a$ component as well as on the $\overline{D_-{}^a}$ component. This implies that both $e_-(s)$ and $e_+(s)$ are represented by exponential functions

$$\mathscr{I}e_-(s) = e^{-izs}n_-\,, \qquad \mathscr{I}e_+(s) = e^{-izs}n_+\,. \tag{3.14}$$

Since W_1 was defined to be the identity on $\overline{D_-{}^a}$, it follows that $e_-(s) = e_0(s)$ for $s < -a$, so that $n_- = n$; this proves (3.12). Now apply W_2 to e; using the definition of S as W_2W_1 and relations (3.3) and (3.2)′ we get

$$\mathscr{I}W_2e = \mathscr{I}e_1 = e^{-izs}\mathscr{S}(z)n.$$

Since by (1.14), $W_2 = I$ on $D_+{}^a$ we conclude that $\mathscr{I}e_+(s) = \mathscr{I}e_1(s)$ for $s > a$, so that $n_+ = \mathscr{S}(z)n$; substituting this into (3.14) we get (3.13); this completes the demonstration of Theorem 3.2.

Being an eigenfunction of the semigroup T, e is also an eigenfunction of the extended A

$$Ae = ize. \tag{3.15}$$

This is easy to show on the basis of the decomposition (2.3) used in Section 2 to define the domain and action of A and T in $\overline{H}$.

The next remark will be useful in Section 10 when we obtain an explicit representation for the scattering matrix. Let J_0 be one of the mappings introduced in Section 1 to take us from the space H_0 to H, so chosen that J_0 carries the domain of A_0 into the domain of A. Since J_0 acts like the identity on the $D_-{}^\rho$ and $D_+{}^\rho$ components, it can be extended without further ado as a mapping from $\overline{H}_0$ to $\overline{H}$. Consider now

$$v = e - J_0e_0\,; \tag{3.16}$$

v is sometimes called the outgoing *scattered wave*. According to (3.12) and (3.13), the $D_-{}^\rho$ component of v is zero and the $D_+{}^\rho$ component of v is represented by $[\mathscr{S}(z)n - n]\,e^{-izs}$. Since $\operatorname{Im} z < 0$, we see that v belongs to H.

We now derive a useful equation for v: Apply $(A - iz)$ to (3.16); then since e itself satisfies (3.15) we get

$$(A - iz)v = (iz - A)\,J_0e_0\,. \tag{3.17}$$

Observe that the right side belongs to the subspace K^ρ. Since the right side is a known quantity, Eq. (3.17) can be used to determine v

for $\operatorname{Im} z < 0$; the D_+^ρ component of v is represented by a function of the form

$$\mathscr{I} v_+ = e^{-izs} m, \tag{3.18}$$

where m is determined by e_0, i.e., by z and n

$$m = \mathscr{K}'(z) n. \tag{3.19}$$

This gives the formula

$$\mathscr{S}(z) = I + \mathscr{K}'(z) \tag{3.20}$$

for the scattering matrix.

4. The Associated Semigroup of Operators and the Analytic Continuation of the Scattering Matrix

We now turn our attention to another tool for the study of the scattering operator, namely a semigroup of operators describing the local behavior of $T(t)$

$$Z(t) = P_+^a T(t) P_-^a, \tag{4.1}$$

where P_+^a and P_-^a are the orthogonal projections on the orthogonal complements of D_+^a and D_-^a, respectively. We choose $a > \rho$.

Theorem 4.1. *The operators $\{Z(t)\}$ form a strongly continuous semigroup of contraction operators on $K^a = H \ominus (D_-^a \oplus D_+^a)$. $Z(t)$ annihilates $D_-^a \oplus D_+^a$.*

Proof. For any f in H we decompose $T(t)f$ into orthogonal parts

$$T(t)f = d(t) + e(t),$$

where $d(t) \in D_+^a$ and $e(t) = P_+^a T(t) f \perp D_+^a$. According to (7 i), $T(s)\, d(t)$ belongs to D_+^a and, hence $P_+^a T(s)\, d(t) = 0$. It follows that

$$P_+^a T(s+t) f = P_+^a T(s) P_+^a T(t) f; \tag{4.2}$$

this is the semigroup property for the operators $\{P_+^a T(t)\}$. Next we show that when $f \perp D_-^a$ so is $P_+^a T(t) f$. Now for $f \perp D_-^a$ and g in D_-^a

$$(P_+^a T(t) f, g) = (T(t) f, P_+^a g) = (T(t) f, g)$$

since we have assumed that $D_-^a \perp D_+^a$. Finally, because of (7 i), $T^*(t)g$ lies in D_-^a and, hence,

$$(P_+^a T(t)f, g) = (f, T^*(t)g) = 0.$$

That $Z(t)$ annihilates D_-^a is obvious; that it annihilates D_+^a follows from the orthogonality of D_+^a to D_-^a and the fact that T maps D_+^a into itself. This completes the proof of Theorem 4.1.

Denote by B the infinitesimal generator of Z. Since each $Z(t)$ is a contraction, the spectrum of B lies in the left half plane

$$\operatorname{Re} \sigma(B) \leqslant 0.$$

According to property (8) in the introduction

$$\lim P_+^a T(t)f = 0$$

for every f in H; this implies that

$$\lim Z(t)f = 0. \tag{4.3}$$

It follows from this that the point spectrum of B contains no purely imaginary values.

It will be helpful to analyze the action of B a little more precisely. Let $f_\pm$ denote the translation representers of the $\overline{D_\pm^\rho}$ components of f in H (or $\bar{H}$). We recall that the action of A on these components is given by

$$\begin{aligned} \mathscr{I}(Af)_+ &= -\partial_s \mathscr{I} f_+ \qquad \text{for} \quad s > \rho, \\ \mathscr{I}(Af)_- &= -\partial_s \mathscr{I} f_- \qquad \text{for} \quad s < -\rho. \end{aligned} \tag{4.4}$$

For f in K^a, $Z(t)f = P_+^a T(t)f$. The effect of P_+^a is simply to chop off $\mathscr{I} f_+$ at $s = a$ and, consequently,

$$\mathscr{I}(Bf)_+ = \begin{cases} -\partial_s \mathscr{I} f_+ & \rho < s < a, \\ 0 & a < s. \end{cases} \tag{4.5}$$

Finally, if g lies in the domain of A and is orthogonal to D_-^a then $P_+^a g$ lies in K^a and $Z(t)\, P_+^a g = P_+^a T(t)g$ as in (4.2). Differentiating with respect to t at $t = 0$, we see that for such g

$$BP_+^a g = P_+^a Ag. \tag{4.6}$$

The next theorem reveals an interesting connection between the semigroup Z and the scattering matrix $\mathscr{S}(z)$.

THEOREM 4.2. *Let $\rho_0(B)$ denote the principal component of the resolvent set of B, i.e., that component of the resolvent set containing the right half plane. Then $\mathscr{S}(z)$ can be continued analytically from the lower half plane into $-i\rho_0(B)$.*

Proof. For any z in the lower half plane and any n in N let e be the element of $\bar{H}'$ defined by (3.10); according to Theorem 3.2, e is an eigenvector of A. We now introduce two cut-off functions, one on $\mathbb{R}_- - \rho$ and the other on $\mathbb{R}_+ + \rho$:

$$\xi_-(s) = \begin{cases} 0 & \text{for} \quad -\rho - \delta < s < -\rho \\ & \text{smooth in between,} \\ 1 & \text{for} \quad s < -a, \end{cases}$$

$$\xi_+(s) = \begin{cases} 0 & \text{for} \quad \rho < s < a, \\ 1 & \text{for} \quad a < s; \end{cases}$$

here $0 < \delta < a - \rho$. We define the element $f = f(z)$ by

$$f = e - \mathscr{I}^{-1}\xi_-\mathscr{I}e_- - \mathscr{I}^{-1}\xi_+\mathscr{I}e_+ , \tag{4.7}$$

where as before e_- and e_+ are the $\overline{D_-^\rho}$ and D_+^ρ components of e. For convenience we will write this as

$$f = e - \xi_-e_- - \xi_+e_+ \tag*{(4.7)$'$}$$

and trust that the reader will be able to keep in mind the role of the translation representation. According to (3.12) and (3.13)

$$e_- = e^{-izs}n \text{ for } s < -\rho \qquad \text{and} \qquad e_+ = e^{-izs}\mathscr{S}(z)n \text{ for } s > \rho. \tag{4.8}$$

Making use of (4.4) and (4.8) we obtain

$$A\xi_-e_- = -\xi_-'e_- + iz\xi_-e_- . \tag{4.9}$$

Now f belongs to K^a and $f + \xi_+e_+$ lies in the domain of A so that (4.6) holds with g replaced by $f + \xi_+e_+$. Combining (4.6), (4.7), and (4.9) with the relation $Ae = ize$ we get

$$\begin{aligned} Bf &= P_+^aA(f + \xi_+e_+) = P_+^aA(e - \xi_-e_-) \\ &= P_+^a(ize + iz\xi_-e_- + \xi_-'e_-). \end{aligned}$$

Subtracting iz times $P_+^a\xi_+e_+ = 0$ from the right side gives

$$Bf = izf + \xi_-'e_- . \tag{4.10}$$

Since Re $iz > 0$, we have $iz \in \rho(B)$ and the relation (4.10) can be solved for f

$$f = -(iz - B)^{-1}(\xi_-' e_-). \tag{4.11}$$

This relation holds for Im $z < 0$ and shows that $f(z)$ can be analytically continued into a set in the upper half plane for which iz remains in the resolvent set of B, that is into $-i\rho_0(B)$.

According to (4.8) e_+ is of the form $e^{-izs}\mathscr{S}(z)n$. It follows from the definition (4.7)′ of f that f_+, the $D_+{}^\rho$ component of f is

$$f_+(s, z) = e^{-izs}\mathscr{S}(z)n \qquad \text{for} \quad \rho < s < a. \tag{4.12}$$

We have already shown that $f(z)$ can be continued analytically into $-i\rho_0(B)$, the same is true of its component f_+, and, therefore, the same is true for $\mathscr{S}(z)n$. This shows that $\mathscr{S}(z)$ has a strong analytic continuation into $-i\rho_0(B)$, as asserted in Theorem 4.2.

The proof of Theorem 4.2 can be run backwards, as follows: Suppose z belongs to $-i\rho_0(B)$, and n is any element of N. Take e_- to be

$$e_- = e^{-izs}n, \qquad s < -\rho,$$

and $\xi_\pm$ as defined previously. Define f to be the solution (4.11). Define e_+ to be

$$e_+ = e^{-izs}\mathscr{S}(z)n,$$

and solve for e from (4.7)′

$$e = f + \xi_- e_- + \xi_+ e_+ . \tag{4.13}$$

Since $\xi_+ e_+$ is a smooth continuation of f it is clear that e belongs to the domain of A. Moreover, it follows from the proof of Theorem 4.2 that for Im $z < 0$ the element e is an eigenvalue of A, i.e.,

$$Ae - ize \tag{4.14}$$

is zero. On the other hand, by construction, the $\overline{D_-{}^a}$ and $\overline{D_+{}^a}$ components of e are exponential and hence it follows that the $\overline{D_-{}^a}$ and the $\overline{D_+{}^a}$ component of (4.14) are zero.

Denote the element (4.14) by $r(z)$; then as previously observed, $r(z)$ belongs to H and is zero when Im $z < 0$. The explicit form of e shows that r depends analytically on z; it follows, therefore, that $r(z)$ has to be zero everywhere. We formulated what we prove in the following theorem.

THEOREM 4.3. *For any z in $-i\rho_0(B)$ and any n in N there is an eigenvector e of A in $\overline{H}$*

$$Ae = ize \tag{4.15}$$

whose $\overline{D_-^\rho}$ and $\overline{D_+^\rho}$ components are given by

$$e_- = e^{-izs}n, \qquad e_+ = e^{-izs}\mathscr{S}(z)n. \tag{4.16}$$

Remark. Under the additional assumptions made in Section 5, e is uniquely determined by (4.15) and (4.16).

A related result which will be needed in Section 10 is the following theorem.

THEOREM 4.4. *For g in K^ρ and any z in $-i\rho(B)$ there exists a solution in $\overline{H}$ to*

$$(iz - A)h = g, \tag{4.17}$$

which is outgoing in the sense of having zero $\overline{D_-^a}$ component.

Remark. Under the additional assumptions made in Section 5, this solution will be unique.

Proof. Let f be the solution to

$$(iz - B)f = g, \tag{4.18}$$

and let f_+ denote the D_+^ρ component of f as before. Since f is in K^a it is automatically orthogonal to D_-^a. Since $g_+ = 0$ we see by (4.5) that

$$(iz + \partial_s)f_+ = 0 \qquad \text{for} \quad \rho < s < a.$$

Consequently, f_+ is of the form

$$f_+ = \begin{cases} e^{-izs}m & \text{for} \quad \rho < s < a, \\ 0 & \text{for} \quad a < s. \end{cases}$$

We now define

$$h_+ = e^{-izs}m \qquad \text{for} \quad s > \rho.$$

We shall show that

$$h = f - f_+ + h_+$$

is a solution of (4.17); it is clear that h is orthogonal to D_-^a. We begin by constructing a smooth function ξ

$$\xi(s) = \begin{cases} 0 & \text{for} \quad s < \rho + \delta, \\ 1 & \text{for} \quad s > a - \delta, \end{cases}$$

where $\delta < (a - \rho)/4$. In this case we can write

$$h = f - \xi f_+ + \xi h_+ . \tag{4.19}$$

Moreover, since ξf_+ obviously lies in the domain of B so does $f - \xi f_+$. Now $f - \xi f_+ = 0$ for $s > a - \delta$ so that

$$T(t)(f - \xi f_+) = Z(t)(f - \xi f_+)$$

for $t < \delta$. Differentiating at $t = 0$ we get

$$A(f - \xi f_+) = B(f - \xi f_+).$$

Combining this with (4.19) we obtain

$$A(h - \xi h_+) = B(f - \xi f_+). \tag{4.20}$$

According to (4.4) and (4.5) we can write

$$B(\xi f_+) = -\xi' f_+ - \xi \partial_s f_+ = -\xi' f_+ + iz\xi f_+ ,$$
$$A(\xi h_+) = -\xi' h_+ - \xi \partial_s h_+ = -\xi' h_+ + iz\xi h_+ .$$

Since the support of ξ' lies in $\rho + \delta < s < a - \delta$ where f_+ and h_+ are the same, we have $\xi' f_+ = \xi' h_+$. Thus,

$$A\xi h_+ - B\xi f_+ = iz(\xi h_+ - \xi f_+) = iz(h - f)$$

by (4.19). Combining this with (4.18) and (4.20) we finally get

$$\begin{aligned} g &= izf - Bf = izf - B(f - \xi f_+) - B(\xi f_+) \\ &= izf - A(h - \xi h_+) - A(\xi h_+) + iz(h - f), \end{aligned}$$

so that

$$g = (iz - A)h.$$

This completes the proof of Theorem 4.4.

Remark. If g belongs to K^ρ then it is not difficult to show that f is orthogonal to D_-^ρ so that the h constructed in Theorem 4.4 is actually orthogonal to D_-^ρ.

We conclude this section with a brief discussion of the spectral mapping theorem for Z and one of its consequences. Let $m(dt)$ be any complex measure on Borel subsets of the nonnegative reals with finite total measure. The Laplace transform of m

$$g(\mu) = \int_0^\infty e^{\mu t} m(dt), \qquad \operatorname{Re} \mu \leqslant 0, \tag{4.21}$$

will be analytic in the open half-plane and continuous in the closed half-plane. For functions g of this kind we define $g(B)$ as

$$g(B) = \int_0^\infty Z(t)\, m(dt). \tag{4.22}$$

The spectral mapping theorem for this set-up is well known (see [3, Theorem 16.6.1]).

THEOREM 4.5. $\sigma(g(B)) \supset g(\sigma(B))$.

THEOREM 4.6. *Suppose that g is a function of the form* (4.21) *and that $g(B)$ is compact. If $g(\mu)$ has no zero in the spectrum of B, then B has a pure point spectrum and the resolvent of B, in symbols $R_\mu(B)$, is meromorphic in the whole complex plane.*

As we have previously noted the point spectrum of B contains no point on the imaginary axis, and, therefore, if B satisfies the hypothesis of this theorem we have the following corollary.

COROLLARY 4.7. $\operatorname{Re} \sigma(B) < 0$.

Proof of Theorem 4.6. According to Theorem 4.5 the spectrum of B is contained in the set of roots of $\lambda = g(\mu)$ for λ in $\sigma(g(B))$. Since 0 is not in $g(\sigma(B))$, it suffices to consider $\lambda \neq 0$ in $\sigma(g(B))$. For such a λ we define

$$E_\lambda = \frac{1}{2\pi i} \oint R_\zeta(g(B))\, d\zeta, \tag{4.23}$$

where the path of integration is a small circle about λ containing no other spectral point of $g(B)$. According to Lemma 5.7.2 [3] the range L of E_λ will be finite dimensional; let M denote the range of $I - E_\lambda$. According to Theorem 5.9.3 [3], $R_\zeta(g(B))\, E_\lambda$ ($= R_\zeta(g(B)\, E_\lambda)$ on L) can be extended to be regular except at $\zeta = \lambda$, whereas $R_\zeta(g(B))(I - E_\lambda)$ can be extended to be regular at $\zeta = \lambda$. It is easy to see that E_λ

commutes with $Z(t)$ and it follows from this that $Z(t)$ leaves L and M invariant.

Because L is finite dimensional $R_\zeta(B)\, E_\lambda = R_\zeta(E_\lambda B)\, E_\lambda$ is meromorphic with a finite set of poles at say $\mu_1, \mu_2, \ldots, \mu_k$. These points are obviously in the point spectra of B and BE_λ. Note also that $g(B)\, E_\lambda = g(BE_\lambda)$ on L has λ as its only spectral point. Applying Theorem 4.5 to $Z(t)$ restricted to L we see that $g(\mu_i) = \lambda$ for $i = 1, 2, \ldots, k$. Since $g(B)$ restricted to M does not have λ in its spectrum, if we apply Theorem 4.5 to $Z(t)$ restricted to M we see that $g(\mu) = \lambda$ has no roots on the spectrum of B restricted to M. In other words, $R_\mu(B)(I - E_\lambda)$ can be extended to be regular at all of the roots of $g(\mu) = \lambda$. Combining these two results we see that

$$R_\mu(B) = R_\mu(B)\, E_\lambda + R_\mu(B)(I - E_\lambda)$$

is regular at all but a finite number of roots of $g(\mu) = \lambda$ and there its singularities are poles. Finally, if a sequence μ_j belonging to $\sigma(B)$ converged to μ_0 then μ_0 lies in $\sigma(B)$ and since g is analytic on the spectrum of B by Corollary 4.7 (we have already established that B has a pure point spectrum), an infinite subset of the $g(\mu_j)$ are distinct; thus, $g(\mu_0)$ is a point of accumulation of the spectrum of $g(B)$ and since $g(B)$ is compact this implies that $g(\mu_0) = 0$; contrary to our assumption on g.

Remark. It follows from the proof of Theorem 4.6 that the generalized eigenspace of B at $\mu \in \sigma(B)$ is of finite dimension.

COROLLARY 4.8. *If $Z(T)\, R_\kappa(B)$ is compact for some $\kappa > 0$, then $R_\mu(B)$ is meromorphic in the whole complex plane.*

Proof. We apply the above theorem to

$$m(dt) = e^{\kappa(T-t)}\, dt \text{ for } t \geqslant T \qquad \text{and} \qquad = 0 \text{ for } 0 < t < T.$$

In this case $g(\mu) = e^{\mu T}(\kappa - \mu)^{-1}$ and $g(B) = Z(T)\, R_\kappa(B)$.

5. THE POLES OF THE SCATTERING MATRIX AND OUTGOING EIGENVECTORS

In this section we assume that *the resolvent of B is meromorphic in the whole plane.* In this case, the *spectrum of B will be pure point spectrum.* We note that at the end of Section 4 we have given a sufficient condition for this to be so; we shall show in Part II that for the application considered there this condition is satisfied.

Under the above assumption $\rho_0(B)$ is all of the resolvent set of B, i.e., the whole plane minus the eigenvalues of B. According to Theorem 4.2, $\mathscr{S}(z)$ can be continued analytically into $-i\rho(B)$; this shows that the singularities of $\mathscr{S}(z)$ are isolated, actually from the proof of Theorem 4.2 a little more follows: *The singularities of $\mathscr{S}$ are poles.* For as iz approaches an eigenvalue of B, $(iz - B)^{-1}$ increases as some power of the distance of iz to that eigenvalue; it follows from (4.11) that the same is true of f and, therefore, of f_+, in terms of which $\mathscr{S}(z)$ is defined.

According to Theorem 4.3, for iz not in $\sigma(B)$ and any n in N, the operator A in $\bar{H}$ has an eigenvector e whose $\overline{D_-^\rho}$ and $\overline{D_+^\rho}$ components are given by (4.16). Suppose z_0 is a zero of $\mathscr{S}(z)$, i.e., a value where the equation

$$\mathscr{S}(z_0)n = 0$$

has a nontrivial solution. For such a z_0 and n the $\overline{D_+^\rho}$ component of the eigenvector e is zero. An eigenvector of this kind is called an *incoming eigenvector*, since its outgoing (i.e., D_+^ρ) component is zero. An *outgoing eigenvector* is defined analogously as one whose D_-^ρ component is zero. We shall show that just as the incoming eigenvectors are associated with zeros of $\mathscr{S}$, so the outgoing ones are associated with the poles of $\mathscr{S}$.

LEMMA 5.1. *Suppose b is an eigenvector of B; then b_-, the D_-^ρ component of b, is zero.*

Proof. An eigenvector of B is also an eigenvector of Z

$$Z(t)b = e^{izt}b. \tag{5.1}$$

Let b_- denote the D_-^ρ component of b; form the scalar product of (5.1) with b_-

$$e^{izt}(b, b_-) = (Z(t)b, b_-) = (T(t)b, b_-) = (b, T^*(t)\, b_-). \tag{5.2}$$

In the third step we have used the fact that since b_- is orthogonal to D_+^a, $P_+^a b_- = b_-$.

Since b_- belongs to D_-^ρ, $T^*(t)\, b_-$ belongs to $D_-^{\rho+t}$; for $t > a - \rho$ this space is orthogonal to K^a. Since b belongs to K^a, (5.2) is zero for $t > a - \rho$, and in particular

$$0 = (b, b_-) = (b_-, b_-),$$

so that $b_- = 0$, as asserted.

LEMMA 5.2. *If iz_0 is an eigenvalue of B, then A has an outgoing igenvector with eigenvalue iz_0, and conversely.*

Proof. Let b be the eigenvector of B; it follows that the $D_+{}^\rho$ component b_+ of b is of the form

$$b_+ = \begin{cases} e^{-izs}m & \rho < s < a, \\ 0 & a < s. \end{cases}$$

Here we continue the practice of not distinguishing between data in $D_\pm{}^\rho$ and their translation representers. We define b_{++} to be

$$b_{++} = \begin{cases} 0 & \rho < s < a, \\ e^{-izs}m & a < s. \end{cases}$$

We claim that

$$a = b + b_{++} \tag{5.3}$$

is an eigenvector of A; we leave the easy verification of this fact to the reader. Clearly, $a_- = b_-$ and, according to Lemma 5.1, $b_- = 0$; this shows that a is indeed outgoing.

To prove the converse we construct b out of a by reversing (5.3), i.e., setting

$$b = a - \zeta_+ a_+ = P_+{}^a a;$$

recall that the $D_-{}^\rho$ component of a is zero. It is easy to show (see (4.6)) that b is an eigenvector of B with eigenvalue iz_0. This completes the proof of Lemma 5.2.

It follows from formula (5.3) relating outgoing eigenvectors of A and eigenvectors of B that if an eigenvector b of B has zero $D_+{}^\rho$ component, then b itself is an eigenvector of A, and, furthermore, b is both outgoing and incoming. We now impose the additional hypothesis (happily satisfied in Part II) that this cannot happen: *Neither A nor A^* have eigenvectors which are both incoming and outgoing*. This assumption has the desired consequence.

LEMMA 5.3. *No eigenvector b of B has a zero $D_+{}^\rho$ component, and no eigenvector b^* of B^* has zero $D_-{}^\rho$ component.*

We saw earlier in Theorem 4.2 that every point in $-i\rho(B)$ is a regular point for $\mathcal{S}$. Under the additional assumption just introduced we can prove the converse.

LEMMA 5.4. *If iz_0 is an eigenvalue of B, then z_0 is a pole of $\mathcal{S}$.*

Proof. We will reverse the proof of Theorem 4.2; the thrust of that argument was: If iz_0 is not an eigenvalue of B then f, defined by (4.7)', is analytic in z near z_0, having at most a pole at z_0. From this it followed that $f_+ = e^{izs}\mathscr{S}(z)n$ for $\rho < s < a$ is also an analytic function of z. Our first step now is to show that if iz_0 is an eigenvalue of B then f, defined by (4.7)', has a pole at z_0.

By assumption the resolvent of B has a pole at iz_0, and it follows from this that the resolvent of B^* has a pole at $-i\bar{z}_0$. As a consequence $-i\bar{z}_0$ is an eigenvalue of B^*; let b^* be a corresponding eigenvector

$$(B - iz_0)^* \, b^* = 0. \tag{5.4}$$

According to Lemma 5.3, the D_-^ρ component b_-^* of b^* is a nonzero exponential on $-a < s < -\rho$; from this we deduce easily that there exists a vector n and a cut-off function ξ_- such that

$$(\xi_-'(s)\, e^{-iz_0 s} n,\, b_-^*) \neq 0. \tag{5.5}$$

Let z be near z_0 and $\neq z_0$. Denote by $e(z)$ the eigenvector of A, described in Theorem 4.3, obtained by analytic continuation of $W_1 e_0(z)$ from the lower half plane. Define $f = f(z)$ by formula (4.7)', then following the steps of the proof of Theorem 4.2, we find that f satisfies (4.10)

$$(iz - B)f = -\xi_-' e_-(z). \tag{5.6}$$

Form the scalar product with b^* and use the fact the $(B - iz_0)^* b^* = 0$; we get

$$i(z_0 - z)(f, b^*) = (\xi_-' e_-(z), b^*) \tag{5.7}$$

according to (4.16),

$$e_-(z) = e^{-izs} n;$$

since e_- belongs to D_-^ρ, it follows from (5.5) that the right side of (5.7) is $\neq 0$. But this implies that $\|f(z)\|$ tends to ∞ as z tends to z_0. This proves that f has a pole at z_0, say of order $k_0 > 0$.

Suppose now that $\mathscr{S}(z)n$ were holomorphic at z_0 and define the Laurent coefficients

$$f_k = \frac{1}{2\pi i} \oint f(z)(z - z_0)^{k-1}\, dz, \qquad k > 0, \tag{5.8}$$

where the path of integration is a sufficiently small circle about z_0. Then

$$Bf_k = \frac{1}{2\pi i} \oint Bf(z)(z - z_0)^{k-1}\, dz,$$

and making use of (4.10) $Bf = izf + \xi_-'e_-$, we get

$$Bf_k = iz_0 f_k + if_{k+1}. \tag{5.9}$$

Note that the $\xi_-'e_-$ term is holomorphic at z_0 and, hence, does not contribute to (5.9). Since z_0 is a pole of order k_0 we see that $f_{k_0+1} = 0$ so that f_{k_0} is an eigenvector of B. Moreover, $f_-(z) = e^{-izs}\mathscr{S}(z)n$ for $\rho < s < a$, and, hence, if $\mathscr{S}(z)n$ were holomorphic at z_0 we could deduce from (5.8) that $(f_{k_0})_+ = 0$. However this would contradict Lemma 5.3, this proves that $\mathscr{S}(z)n$ is singular at z_0 and completes our proof of Lemma 5.4.

Remark. The above argument shows that $f(z)$ has a removable singularity at z_0 whenever $\mathscr{S}(z)n$ is holomorphic at z_0. Defining $f(z_0)$ by analytic continuation, we see that $Bf(z_0) = iz_0 f(z_0) + \xi_-'e_-$, and, hence, we can repeat the proof of Theorem 4.3 to obtain an eigenvector e of A satisfying (4.16) for $z = z_0$ and this particular n.

We can now state the main result of this section.

THEOREM 5.5. *Under the additional hypotheses imposed in this section*:

(I) *The resolvent of B is meromorphic in the whole complex plane.*

(II) *Neither A nor A^* have eigenvectors which are both incoming and outgoing,*

the following conditions are equivalent:

(1) *The scattering matrix $\mathscr{S}(z)$ has a pole at z_0;*

(2) *iz_0 belongs to the spectrum of B;*

(3) *A has an outgoing eigenvector with eigenvalue iz_0.*

Proof. The equivalence of (2) and (3) is Lemma 5.2. That (1) implies (2) is Theorem 4.2, and that (2) implies (1) is Lemma 5.4. This completes the proof of Theorem 5.5.

We turn now to the incoming eigenvectors of A. If iz is not an eigenvalue of B then Theorem 4.3 describes a class of eigenvectors $e(z)$ of A with eigenvalue iz; they are the ones obtained by analytic continuation of $W_1 e_0$ from the lower half plane; recall that $e_0 = e^{-izs}n$. We claim that these are all the eigenvectors for such an eigenvalue. For suppose there were another one g; being an eigenvector the $\overline{D_-^\rho}$

component of g is of the form $e^{-izs}n$. According to Theorem 4.3, there is an eigenvector e whose $\overline{D_-{}^\rho}$ component has the same form. Therefore, $g - e$ would be an outgoing eigenvector, but according to Lemma 5.2 this cannot happen when iz is not an eigenvalue of B.

An eigenvector $e(z)$ of A described by Theorem 4.3 has $D_+{}^\rho$ component $e^{-izs}\mathscr{S}(z)n$; therefore, *$e(z)$ is incoming if and only if $\mathscr{S}(z)n = 0$.* This characterizes all incoming eigenvectors of A with eigenvalue iz, where z is a regular point of $\mathscr{S}$. There may, however, be additional incoming eigenvectors at poles of $\mathscr{S}$. For instance, if for a particular n the function $\mathscr{S}(z)n$ happens to be regular at a pole z_0 of $\mathscr{S}$, then we claim that the eigenfunction $e(z)$ also is regular at z_0. This follows from the remark preceding Theorem 5.5.

If $\mathscr{S}(z)n$ is not only regular at z_0 but vanishes there, then the eigenfunction $e(z_0)$ is incoming. However, we were not able to show that these are all the incoming eigenfunctions of A associated with poles of $\mathscr{S}$.

Our interest in incoming eigenfunctions for $\operatorname{Im} z > 0$ is that they are the only ones which belong to H; that is, they are eigenfunctions not only of the extended operators $T(t)$ and A but of the original operators over H.

THEOREM 5.6. *The point spectrum of A over H is of the form iz, where* $\operatorname{Im} z > 0$ *and z is a zero of $\mathscr{S}(z)$ or possible a pole.*

6. THE SPECTRUM OF A

In the previous section we characterized the point spectrum of A; it is contained in the left half plane $\operatorname{Re} \lambda < 0$ and is in one-to-one correspondence with the zeros of the scattering matrix $\mathscr{S}(z)$ in the upper half plane and possibly with some of the poles of $\mathscr{S}$, the correspondence being given by $\lambda = iz$. How completely does this characterize the spectrum of A? We shall investigate this question in the present section.

THEOREM 6.1. *The imaginary axis is contained in the continuous part of the spectrum of A.*

Proof. The restriction of $T(t)$ to $D_+{}^\rho$ corresponds in the translation representation to translation on a half-line. It is well known that the spectrum of the generator of such a semigroup of operators fills out the entire half-plane $\operatorname{Re} \lambda \leqslant 0$. Any extension of this generator such as A will have in its spectrum at least the boundary of this set (see Theorem 4.11.2 of [3])—that is the imaginary axis. Neither A nor A^* can have purely imaginary eigenvalues for the corresponding eigenvector would have to be both incoming and outgoing, and, therefore, the points on the imaginary axis must be continuous spectral points for A.

We now give a criterion for the rest of the spectrum of A to be point spectrum.

THEOREM 6.2. *Suppose that there is a skew-self-adjoint operator* A_1 *on* H *such that*

$$R_{\lambda_0}(A) - R_{\lambda_0}(A_1)$$

is compact for some λ_0. *Then that portion of the spectrum of* A *in the half plane* $\operatorname{Re}\lambda < 0$ *is entirely point spectrum and is either discrete in this half plane or fills out the entire half plane.*

Remark. Both of these possibilities can occur. In fact the hypothesis of Theorem 6.2 is satisfied by both of the examples given in the introduction. In Example 1 the point spectrum fills out the entire half plane, whereas in Example 2 the point spectrum is discrete.

Proof. We recall that when λ_0 belongs to $\rho(A)$, then $\lambda \in \rho(A)$ if and only if

$$I + (\lambda - \lambda_0)\, R_{\lambda_0}(A)$$

is one-to-one and has a bounded inverse (see, [3, p. 187]). Since A_1 is skew-self-adjoint any λ with $\operatorname{Re}\lambda \neq 0$ lies in $\rho(A_1)$ so that for fixed λ_0 with $\operatorname{Re}\lambda_0 > 0$

$$M_\lambda \equiv I + (\lambda - \lambda_0)\, R_{\lambda_0}(A_1)$$

has a bounded holomorphic inverse for all such λ. We now write

$$I + (\lambda - \lambda_0)\, R_{\lambda_0}(A) = I + (\lambda - \lambda_0)\, R_{\lambda_0}(A_1) + (\lambda - \lambda_0)[R_{\lambda_0}(A) - R_{\lambda_0}(A_1)]$$

and multiply through on the right by M_λ^{-1}. This gives

$$M_\lambda^{-1}[I + (\lambda - \lambda_0)\, R_{\lambda_0}(A)] = I + (\lambda - \lambda_0)\, M_\lambda^{-1}[R_{\lambda_0}(A) - R_{\lambda_0}(A_1)].$$

By assumption the expression on the right is of the form I plus a compact operator-valued function holomorphic in the left half plane $\operatorname{Re}\lambda < 0$. It is well known [see [12]) that this expression has a meromorphic inverse if it is invertible at one point in this half-plane.

At those λ for which the inverse exists it is clear that $I + (\lambda - \lambda_0)R_{\lambda_0}(A)$ is also invertible so that $\lambda \in \rho(A)$. At those λ for which the inverse does not exist the compactness assumption assures us of nontrivial null vectors for $I + (\lambda - \lambda_0) R_{\lambda_0}(A)$ and it is easy to show that such a null vector is an eigenvector for A at λ.

Remark. It often happens that $\mathscr{S}(z)$ is of the form

$$\mathscr{S}(z) = I + \mathscr{K}'(z),$$

where $\mathscr{K}'$ is a compact operator on N. In this case if $\mathscr{S}(z)$ is invertible for some z_0 then $\mathscr{S}^{-1}(z)$ will be meromorphic on $-i\rho(B)$. In particular $\mathscr{S}(z)$ will be invertible for some z in the upper half plane and we may conclude from Theorems 5.6 and 6.2 that the point spectrum of A will be discrete in the upper half plane provided of course that the hypothesis of Theorem 6.2 is satisfied.

II. THE ACOUSTIC EQUATION IN AN EXTERIOR DOMAIN WITH DISSIPATIVE BOUNDARY CONDITIONS

Part I of this paper was abstracted from our experience with the acoustic equation in an exterior domain with dissipative boundary conditions. It should, therefore, come as no surprise to find that the assumptions introduced in Part I are satisfied by this system. The purpose of Part II is to show that this is so; in the process we obtain concrete instances of the abstract concepts introduced in Part I.

7. Translation Representations for the Solution of the Wave Equation in Free Space

The solutions of the wave equation

$$u_{tt} = \Delta u \qquad \text{in} \quad R^n \tag{7.1}$$

are uniquely determined by their initial data

$$d = \{u(x, 0), u_t(x, 0)\}; \tag{7.2}$$

the energy

$$\int [| \partial_x u |^2 + | \partial_t u |^2] \, dx$$

is independent of time. Consequently, a natural setting for this problem is the Hilbert space H_0 of all initial data d with finite energy and norm defined as in (10).

We denote by $U_0(t)$ the operator mapping initial data at time 0 into data at time t. It is not difficult to prove that $U_0(t)$ defines a one-parameter group of unitary operators on H_0 with infinitesimal generator A_0 of the form

$$A_0 = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}. \tag{7.3}$$

The domain of A_0 consists of all pairs of functions $\{f_1, f_2\}$ such that $\partial_x^{\beta} f_1$ and $\partial_x^{\beta'} f_2$ belong to $L_2(R^n)$ for all $|\beta| = 1, 2$ and $|\beta'| = 0, 1$. The incoming and outgoing subspaces D_-^{ρ} and D_+^{ρ} are defined as the initial data of all solutions which vanish in the truncated backward or forward cone $\{|x| \leqslant \rho - t, t \leqslant 0\}$ or $\{|x| < \rho + t, t \geqslant 0\}$, respectively.

The simplest way of studying the wave equation in free space is by means of the Fourier transform; perhaps the most penetrating way is by means of the *Radon transform*. We now sketch the Radon transform approach, the full details can be found for n odd in [6] and for n even in [7].

To begin with let f be a smooth, rapidly decreasing function defined in $\mathbb{R}^n$. Its Radon transform, denoted by $\tilde{f}$, is a function of (s, ω) in $\mathbb{R} \times S^{n-1}$ defined as

$$\tilde{f}(s, \omega) = (2\pi)^{(1-n)/2} \int_{x \cdot \omega = s} f(x)\, dS. \tag{7.4}$$

We list the principal properties of the Radon transforms:

(i) If $f(x) = 0$ for $|x| > R$, then

$$\tilde{f}(s, \omega) = 0 \qquad \text{for} \quad |s| > R;$$

(ii) $\tilde{f}$ is an even function of (s, ω);

(iii)
$$\widetilde{(\partial_{x_i} f)} = \omega_i \partial_s \tilde{f}. \tag{7.6}$$

(iv) The following Plancheral formula holds:

$$\int |f(x)|^2\, dx = \tfrac{1}{2} \iint |\partial_s^{(n-1)/2} \tilde{f}|^2\, ds\, d\omega. \tag{7.7}$$

We define the Radon transform of the inital data $d = \{f_1, f_2\}$ *as*

$$Rd = \partial_s \tilde{f}_1 - \tilde{f}_2 . \tag{7.8}$$

It is easy to verify that

$$RA_0 d = R\{f_2, \Delta f_1\} = \partial_s \tilde{f}_2 - \partial_s^2 \tilde{f}_1 = -\partial_s Rd \tag{7.9}$$

so that R gives a translation representation for U_0; however, R is not the representation described in the introduction since it is not unitary.

To rectify this deficiency we first note that the energy norm can be expressed in terms of Rd as

$$\| d \|_E^2 = \tfrac{1}{2} \| Rd \|_{(n-1)/2}^2, \tag{7.10}$$

where for a function $k(s, \omega)$ the norm on the right is the Sobolev norm

$$\| k \|_\nu^2 = \int | \sigma |^{2\nu} | Fk |^2 \, d\sigma \, d\omega;$$

here F denotes the one-dimensional Fourier transform with respect to s. For n odd it follows that

$$\mathscr{I} d \equiv 1/\sqrt{2}\, \partial_s^{(n-1)/2} Rd \tag{7.11}$$

defines a unitary translation representation for U_0 onto $L_2(R, N)$ where $N = L_2(S^{n-1})$, and it can be shown that $\mathscr{I}$ maps D_-^ρ onto $L_2((-\infty, -\rho), N)$ and D_+^ρ onto $L_2((\rho, \infty), N)$.

For n even we have to deal with the half-order derivatives $J_\pm$; as we shall see the two obvious choices work. The simplest way to describe J_- and J_+ is in terms of their action on FRd. Setting

$$\mathscr{I}_-(\sigma) = \begin{cases} \sigma^{1/2} \sigma^{n/2-1} & \text{for } \sigma > 0, \\ i \,| \sigma |^{1/2} \sigma^{n/2-1} & \text{for } \sigma < 0, \end{cases} \tag{7.12$_-$}$$

$$\mathscr{I}_+(\sigma) = \begin{cases} \sigma^{1/2} \sigma^{n/2-1} & \text{for } \sigma > 0, \\ -i \,| \sigma |^{1/2} \sigma^{n/2-1} & \text{for } \sigma < 0, \end{cases} \tag{7.12$_+$}$$

we define

$$\mathscr{I}_\pm d = 1/\sqrt{2}\, J_\pm Rd \equiv 1/\sqrt{2} (F^{-1} \mathscr{I}_\pm F) Rd. \tag{7.13}$$

It is not hard to show that $\mathscr{I}_- D_-^\rho = L_2((-\infty, -\rho), N)$ and that $\mathscr{I}_+ D_+^\rho = L_2((\rho, \infty), N)$.

In terms of the Radon transform the solution of the wave equation with initial data d has the elegant form

$$u(x, t) = \int h(x \cdot \omega - t, \omega) \, d\omega, \tag{7.14}$$

where for n odd

$$h = c_1 \partial_s^{n-2} Rd, \qquad c_1 = \tfrac{1}{2}(-1/2\pi)^{(n-1)/2} \tag{7.15$_{\text{odd}}$}$$

and for n even

$$h = c_1 K \partial_s^{n-2} Rd; \tag{7.15$_{\text{even}}$}$$

here K denotes the Hilbert transform.

For n odd, combining (7.11) and (7.15) gives

$$h = \sqrt{2}\, c_1 \partial_s^{(n-3)/2} \mathscr{I} d. \tag{7.16}$$

This formula shows that if $\mathscr{I} d$ is supported in $(-\infty, -\rho)$ or (ρ, ∞), so is h. Formula (7.14) shows that the solution u with such initial data d vanishes in the backward cone $\{|x| < \rho - t\}$ or forward cone $\{|x| < \rho + t\}$, respectively.

The initial data of the solution u defined by (7.14) are

$$u(x, 0) = \int h(x \cdot \omega, \omega)\, d\omega, \qquad u_t(x, 0) = -\int \partial_s h(x \cdot \omega, \omega)\, d\omega. \tag{7.17}$$

Formulas (7.16) and (7.17) define the inverse of the translation representation $\mathscr{I}$.

We turn next to the extension $\bar{H}_0$ of H_0 for n odd. By definition, elements d of $\bar{H}_0$ are represented in the translation representation by locally L_2 functions $\mathscr{I} d$. Now h as defined by (7.16) depends locally on $\mathscr{I} d$; formulas (7.17) show that values of $u(x, 0)$ and $u_t(x, 0)$ in $|x| < R$ do not depend on values of $h(s, \omega)$ for $|s| > R$. Therefore, it follows for n odd that if $\mathscr{I} d$ is locally L_2, *formulas* (7.16) *and* (7.17) *give a concrete realization of the corresponding element of* $\bar{H}_0$ *as Cauchy data* $\{u, u_t\}$, *where* u_{x_i} *and* u_t *are locally* L_2 *in* $\mathbb{R}^n$; we take this to be the definition of the mapping $\mathscr{I}^{-1}$ from the translation representation of $\bar{H}_0$ to data which are locally in H_0.

Suppose that d belongs to the domain of the extended A_0; according to Section 2 this means that $\mathscr{I} d$ and its distribution derivative $\partial_s \mathscr{I} d$ are locally L_2. Setting $d' = A_0 d$ simply means that

$$\mathscr{I} d' = -\partial_s \mathscr{I} d. \tag{7.18}$$

It is clear from (7.16) that the corresponding h distributions are similarly related; $h' = -\partial_s h$ and it follows from (7.17) that the corresponding data are related by

$$\Delta u = \int \partial_s^2 h(x \cdot \omega, \omega)\, d\omega = -\int \partial_s h'(x \cdot \omega, \omega)\, d\omega = u_t',$$

$$u_t = -\int \partial_s h(x \cdot \omega, \omega)\, d\omega = u'.$$

In other words the concrete realizations of d and d' are related by

$$\{u', u_t'\} = A_0\{u, u_t\}, \tag{7.19}$$

where now A_0 is the differential operator (7.3).

Note that not all Cauchy data $\{u, u_t\}$ with u_x and u_t locally L_2 correspond to elements of $\bar{H}_0$; we have not given a criterion for deciding which do nor have we given a formula for obtaining the translation representation of those which do. The only result in this direction which we need is contained in Lemma 10.6.

8. The Wave Equation in the Exterior of an Obstacle

We now study the lossy wave equation in an exterior domain G. Our method permits energy to be dissipated both in the medium and on the boundary. In order to simplify our presentation we shall consider a lossless medium with dissipative boundary conditions of the type

$$u_n + \alpha u_t = 0 \qquad \text{on} \quad \partial G, \qquad \alpha \geqslant 0. \tag{8.1}$$

In this case the Hilbert space H is the set of all initial data $d = \{f_1, f_2\}$ of finite energy in G and the norm is given by

$$\| d \|_E^2 = \int_G [| \partial_x f_1 |^2 + | f_2 |^2] \, dx \tag{8.2}$$

The operator $T(t)$, taking initial data at time 0 into data at time t, is best defined in terms of the infinitesimal generator

$$A = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}, \tag{8.3}$$

whose domain $D(A)$ consists of all pairs $\{f_1, f_2\}$ such that $\partial_x^{\beta} f_1$ and $\partial_x^{\beta'} f_2$ are in $L_2(G)$ for $|\beta| = 1, 2$, $|\beta'| = 0, 1$ and such that $\partial_n f_1 + \alpha f_2 = 0$ on ∂G. It is clear that $D(A)$ is dense in H.

It is easy to show that A is dissipative; in fact

$$(Ad, d)_E + (d, Ad)_E = -2 \int_{\partial G} \alpha \, | f_2 |^2 \, dS \leqslant 0 \tag{8.4}$$

for all $d = \{f_1, f_2\}$ in $D(A)$. We need to prove that A is maximal dissipative, and hence, the generator of a semigroup of contraction operators. For this it suffices to show that the weak extension A_w is equal to A. Proceeding as in [6, pp. 137–139] we can prove that $D(A_w)$ consists of all d in H such that Δf_1 and $\partial_x f_2$ exist in the sense of distributions and are square integrable on G; and $\partial_n f_1 + \alpha f_2 = 0$ on ∂G in the weak sense. It then follows from the theory of elliptic partial differential equations that d actually belongs to $D(A)$, that $A_w d = Ad$, and, furthermore, we have the following theorem.

THEOREM 8.1. *Suppose that ∂G is of class C^2 and $\alpha \in C(\partial G)$. Then for all d in $D(A)$*

$$\int_G [| \partial_x^2 f_1 |^2 + | \partial_x f_2 |^2] \, dx \leqslant \text{const} \| Ad \|_E^2 . \tag{8.5}$$

Since A is dissipative the spectrum of A lies in the left halfplane: $\operatorname{Re} \sigma(A) \leqslant 0$. We now prove three results about the point spectrum of A, the second of which is a verification of assumption II in Theorem 5.5.

LEMMA 8.2. $\lambda = 0$ *is not in the point spectrum of A.*

Proof. An eigenfunction at $\lambda = 0$ would have a vanishing second component and for its first component a harmonic function with square integrable gradient which satisfies a Neumann boundary condition; it follows by potential theory that the first component also vanishes.

THEOREM 8.3. *For n odd, A has no eigenfunctions which are both incoming and outgoing.*

Proof. Suppose that a is such an eigenfunction at λ; by the previous lemma we know that $\lambda \neq 0$. Let j_0 be any C^∞ scalar function which vanishes near ∂G and is equal to one sufficiently far out, say for $| x | > \rho$. Define b to be $j_0 a$ in G and equal to zero outside of G. Then b is defined everywhere, has finite energy, and satisfies

$$(A_0 - \lambda) b = g,$$

where g vanishes in the region where j_0 is identically one, that is for $| x | > \rho$. According to Theorem 4.2, [6, Chapter 4], b and, therefore, a vanish for $| x | > \rho$. Since by elliptic theory the solutions of $Aa = \lambda a$ are analytic, we conclude that $a \equiv 0$.

THEOREM 8.4. *A has no purely imaginary points in its point spectrum.*

For a proof of this see [6] or [4].

We now choose ρ so large that the obstacle is contained in the ball of radius $\rho/2$ about the origin. Since data in $D_\pm{}^\rho$ vanish in this ball we see that $D_-{}^\rho$ and $D_+{}^\rho$ are subspaces of H. Moreover, it is clear that A_0 and A act in the same way on data which vanish in this ball, and, consequently, property (6) is satisfied.

Assumption I of Theorem 5.5 is also valid; this follows directly from Corollary 4.8 and the following theorem.

THEOREM 8.5. *$Z(2\rho)\, R_\kappa(B)$ is a compact operator for $\kappa > 0$.*

The proof of this theorem is precisely the same as that of Theorem 3.1 [6, Chapter 5] and is, therefore, omitted.

Let j_0 be the cut-off function defined earlier in this section. Multiplication by j_0 serves as both operators J_0 and J, carrying H_0 into H and back. Since data of class $D_+{}^\rho$ and $D_-{}^\rho$ vanish for $|x| < \rho$ and since $j_0(x) = 1$ for $|x| > \rho$, the operators J_0 and J defined previously act as the identity on $D_+{}^\rho$, $D_-{}^\rho$.

We also note that the elements of the extended space $\bar{H}$ can be realized as data $\{u, u_t\}$ with $\partial_x u$ and u_t locally L_2. In fact, if we first project out the K^a component (which belongs to H), then the resulting data lies in $\bar{H}_0$, the corresponding h function of (7.16) vanishes for $|s| < a$ and the inversion formula (7.17) furnishes us with data vanishing in $|x| < a$, that is data which is locally in H as is the K^a component.

Finally, if d in $\bar{H}$ is in the domain of the extended A with

$$d' = Ad,$$

we would like to show that this expression can be given the usual interpretation in terms of the above concrete representation. Write $d = d_0 + d_\infty$ where, in the notation of (2.4),

$$\mathscr{I} d_\infty = \zeta_- \mathscr{I} d_- + \zeta_+ \mathscr{I} d_+ .$$

For d to belong to the domain of the extended A means that d_0 lies in the domain of A and that the functions $\partial_s \mathscr{I} d_-$ and $\partial_s \mathscr{I} d_+$ are locally square integrable. As above the data $\mathscr{I}^{-1}\zeta_- \mathscr{I} d_-$ and $\mathscr{I}^{-1}\zeta_+ \mathscr{I} d_+$ vanish for $|x| < a$ and are, therefore, locally in both H_0 and H. Since A and A_0 act in the same way on such data we see by (7.19) that d_∞ lies locally in the domain of A and that $\mathscr{I} A\, d_\infty = -\partial_s \mathscr{I} d_\infty$. It now follows directly from the definition of $A\mathscr{I}d$ given in Section 2 that

$$Ad = d', \tag{8.6}$$

where now A has the usual interpretation of a differential operator acting locally on data which is locally in H.

9. Local Energy Decay

The main result of this section is a verification of property (8) and this in turn is based on local energy decay. Local energy decay was first proved for dissipative hyperbolic systems by Iwasaki [4] who adapted an earlier proof for conservative systems by the authors [6]. In this paper we adapt still another of our proofs to obtain a weak version of the local energy decay. This is enough for our proof of property (8); the stronger form of local energy decay is an obvious consequence of property (8).

Theorem 9.1. *Let G' be a subdomain of G and denote by $E(d, G')$ that part of the energy of data d located in G'*

$$E(d, G') = \int_{G'} [| \partial_x f_1 |^2 + | f_2 |^2] \, dx.$$

For any finite subcollection $g_1, g_2, \ldots, g_k$ of data in H and any bounded subdomain G'

$$\liminf_{t\to\infty} \sum_{i=1}^{k} E(T(t) g_i, G') = 0. \tag{9.1}$$

Proof. The present proof is similar to that described in [6, pp. 145–147]. It obviously suffices to prove (9.1) for g_i's chosen from a dense subset of H; so we can suppose that the g_i's belong to $D(A)$. For such a g

$$\| AT(t)g \|_E = \| T(t) Ag \|_E \leqslant \| Ag \|_E .$$

Therefore, using Theorem 8.1 we can estimate the square integrals of the first and second order space derivatives of the first components of $T(t)g$ as well as the zero and first order space derivatives of the second components in terms of $\| g \|_E^2 + \| Ag \|_E^2$. We deduce from the Rellich compactness theorem that $\{T(t)g, t \geqslant 0\}$ lies in a compact set in the local energy norm. Hence, in order to establish (9.2) it suffices to construct a sequence $\{t_j\} \to \infty$ such that

$$\lim_{j\to\infty}(T(t_j)g_i\,, f) = 0$$

for all f and all g_i. Since the $T(t_j)$ are uniformly bounded, it suffices to show this relation for a dense set $\{f_k\}$ of data f. This is accomplished in [6] for a unitary group $T(t)$ as follows:

According to Stone's theorem, a unitary group $T(t)$ has the spectral representation

$$T(t) = \int e^{i\lambda t}\, dE(\lambda),$$

from which we get

$$(T(t)g_j\,, f_k) = \int e^{i\lambda t}\, dm_{jk}\,,$$

with

$$m_{jk} = (E(\lambda)g_j\,, f_k).$$

If T has no point eigenvalue, it follows that the spectral measure E, and thereby the measures m_{jk}, have no point-mass. We then appeal to a theorem of Wiener, according to which the Fourier transform of a measure without point-masses tends to zero in the mean. From this fact the existence of the desired sequence t_j follows easily.

To extend this line of argument to the dissipative case we need an analog of Stone's representation theorem. Such a representation can be obtained from the Sz. Nagy theory of unitary dilations (see [13]); however for the sake of completeness we will give a direct argument based on the following characterization of $(T(t)f, f)$ as the Fourier transform of a positive measure.

LEMMA 9.2. *Set*

$$T^{\dagger}(t) = T(t) \quad \text{for} \quad t \geqslant 0 \qquad \text{and} \qquad T^{\dagger}(t) = T^{*}(-t) \quad \text{for} \quad t \leqslant 0.$$

Then $(T^{\dagger}(t)f, f)$ can be represented as a Fourier-Stieltjes integral:

$$(T^{\dagger}(t)f, f) = \int_{-\infty}^{\infty} e^{i\varphi t}\, dp,$$

where p is a positive measure depending on f.

Proof. It is clear that for each $\sigma > 0$ the function

$$k(t, \sigma) \equiv e^{-\sigma|t|}(T^\dagger(t)f, f)$$

defines a well-tempered distribution $k(\sigma)$ over the rapidly decreasing functions on $\mathbb{R}$. Moreover $k(\sigma)$ converges strongly to $k(0)$ as $\sigma \to 0+$ and since the Fourier transform F is an isomorphism on the well-tempered distributions we see that $Fk(\sigma)$ converges strongly to $Fk(0)$. We shall prove that $Fk(\sigma)$ corresponds to a positive measure for each $\sigma > 0$ and it will then follow by a well known property of distributions that $Fk(0)$ is likewise of this form, as asserted. Now

$$\begin{aligned} Fk(\sigma) &= \int_{-\infty}^{\infty} e^{i\nu t}e^{-\sigma|t|}(T^\dagger(t)f, f)\,dt \\ &= \int_0^{\infty} (e^{-(\sigma - i\nu)t}T(t)f, f)\,dt \\ &\quad + \int_0^{\infty} (e^{-(\sigma + i\nu)t}T^*(t)f, f)\,dt \\ &= 2\,\mathrm{Re}([(\sigma - i\nu) - A]^{-1}f, f). \end{aligned}$$

Setting

$$g = [(\sigma - i\nu) - A]^{-1}f$$

we can write

$$([(\sigma - i\nu) - A]^{-1}f, f) = (g, [(\sigma - i\nu) - A]g),$$

so that

$$\mathrm{Re}([(\sigma - i\nu) - A]^{-1}f, f) = \sigma(g, g) - \mathrm{Re}(Ag, g).$$

This expression is positive since the generator A of a semigroup of contraction operators is necessarily dissipative. This shows that $Fk(\sigma)$ is a positive function of ν and completes the proof of Lemma 9.2.

By polarization a similar representation holds for the bilinear forms:

$$(T(t)g, f) = \int e^{i\varphi t}\,dm,$$

where the complex measure m depends on g and f.

We claim that none of these measures contains a point-mass; we shall deduce this fact from

LEMMA 9.3. *The limit*

$$\operatorname*{weak\,limit}_{s\to\infty} \frac{1}{s}\int_0^s e^{-i\varphi_0 t}\, T(t)g\, dt \tag{9.2}$$

exists for all g and all φ_0.

Proof. Using the Fourier representation of (Tf, g) we get

$$\frac{1}{s}\int_0^s (T(t)g, f)\, e^{-i\varphi_0 t}\, dt = \int_0^\infty \frac{e^{i(\varphi-\varphi_0)s} - 1}{i(\varphi - \varphi_0)s}\, dm.$$

The integrand equals 1 for $\varphi = \varphi_0$, and for any other value of φ tends to zero as s tends to ∞. Clearly, the limit of the above integral exists, and equals m_0, the mass located at φ_0 of the measure m.

COROLLARY 9.4. *The limit* (9.2) *is different from zero if and only if one of the measures m contains a nonzero mass at φ_0.*

Next we observe that the weak limit, which we denote by a, is an eigenfunction of $T(r)$ with eigenvalue $e^{ir\varphi_0}$:

$$T(r)a = e^{ir\varphi_0}\, a.$$

To see this, we rewrite the left side of (9.2) using the definition of a:

$$\begin{aligned}
T(r)a &= T(r) \operatorname{weak\,lim} \frac{1}{s}\int_0^s e^{-i\varphi_0 t}\, T(t)g\, dt \\
&= \operatorname{weak\,lim} \frac{1}{s}\int_0^s e^{-i\varphi_0 t}\, T(t+r)g\, dt \\
&= \frac{1}{s} e^{i\varphi_0 r} \int_r^{s+r} e^{-i\varphi_0 u}\, T(u)g\, du \\
&= e^{i\varphi_0 r} \frac{1}{s}\int_0^s e^{-i\varphi_0 u}\, T(u)g + \frac{e^{i\varphi_0 r}}{s}\left[\int_s^{s+r} - \int_0^r\right].
\end{aligned}$$

Clearly, the first terms in the last line tends weakly to $e^{i\varphi_0 r}a$, and the second term tends strongly to zero. This proves our contention that a is an eigenfunction of T.

According to the Corollary 9.4, the weak limit a is nonzero only if the measure m in the Fourier representation of $(T(t)g, f)$ contain a point-mass. In this case a is a nontrivial eigenfunction of A with purely imaginary eigenvalue. Since we have shown in Section 8 that no such eigenfunctions exist, it follows that none of the representing measures $m(g, f)$ contain a point-mass. From this point on our argument for the existence of a sequence of t_j tending to ∞ for which

$$\lim_j (T(t_j)g_i, f_k) = 0$$

proceeds as before.

We now come to the proof of property (8).

THEOREM 9.5. st $\lim_{t\to\infty} P_+^{\rho} T(t) = 0$.

Proof. It suffices to establish this result for a dense subset of H, in this case $AD(A^N)$ or $A^2D(A^N)$ for some integer N. That these subsets are dense in H follows from the fact that 0 is not in the point spectrum for either A or A^* by Lemma 8.2. Also recall from (4.2) that $\{P_+T(t)\}$ form a semigroup of contractions so that $\| P_+T(t)g \|_E$ is in any case monotonically nonincreasing in t. Consequently, we need only prove

$$\liminf_{t\to\infty} \| P_+^{\rho} T(t)g \|_E = 0. \tag{9.3}$$

Setting $G_R = G \cap \{| x | < R\}$, we apply Theorem 9.1 to the domain $G_{4\rho}$ and the data $g, Ag,..., A^N g$. Thus, for a given $\epsilon > 0$ there exist arbitrary large τ such that

$$E(T(\tau)\, A^i g;\, G_{4\rho}) < \epsilon, \qquad i = 0, 1,..., N. \tag{9.4}$$

Since energy propagates at a speed $\leqslant 1$ we conclude that

$$E(T(t)d,\, G_\rho) \leqslant E(d,\, G_{4\rho})$$

for all data d and for $0 \leqslant t \leqslant 3\rho$. Applying this to $d = T(\tau)\, A^i g$ we deduce from (9.4) that

$$E(A^i T(t)g,\, G_\rho) < \epsilon \quad \text{for} \quad \tau \leqslant t \leqslant \tau + 3\rho \quad \text{and} \quad 0, 1,..., N. \tag{9.5}$$

We denote the first component of $T(t)\, Ag$ by $w = w(x, t)$ and choose $\xi \in C^\infty(R^n)$ so that $\xi(x)$ vanishes in a neighborhood of ∂G and is identically one for $| x | > \rho/2$. We then set

$$v = \xi w;$$

here we have used Ag in the definition of w instead of g itself in order to obtain a direct $L_2(G_\rho)$ estimate on w. If we extend v to be zero within the obstacle, it then satisfies the inhomogeneous wave equation

$$\begin{gathered} v_{tt} - \Delta v = q \qquad \text{on} \quad \mathbb{R}^n, \\ v(x, 0) = \xi g_2, \qquad v_t(x, 0) = \xi \Delta g_1 ; \end{gathered} \tag{9.6}$$

here

$$q = -w\Delta\xi - 2\partial_x w \cdot \partial_x \xi.$$

Recall that P_+^ρ acts like the identity on data with support in the ball $\{|x| < \rho\}$. Hence, making use of (9.5) we see that for $\tau \leqslant t \leqslant \tau + 3\rho$

$$\begin{aligned} \| P_+^\rho T(t) Ag \|_E &\leqslant \| P_+^\rho\{v, v_t\}\|_E + \left[\int [|\partial_x(1 - \xi)w|^2 + |(1 - \xi) w_t|^2] \, dx\right]^{1/2} \\ &\leqslant \| P_+^\rho\{v, v_t\}\|_E + O(\epsilon^{1/2}). \end{aligned}$$

It, therefore, suffices to obtain a suitable estimate for $P_+^\rho\{v, v_t\}$. Since v is the solution to a free space problem we can now employ the (outgoing) translation representation of U_0; note that D_+^ρ maps onto $L_2((\rho, \infty), N)$ in this representation. It will be convenient to employ the distribution valued Radon transform R as a book keeping device; estimates will always be made on suitable derivatives which belong to $L_2(\mathbb{R}, N)$.

If we set

$$m = R\{v, v_t\} \qquad \text{and} \qquad l = R\{0, q\}$$

then the Radon transform of (9.6) is simply

$$(\partial_t + \partial_s)m = l.$$

Abbreviating $\partial_s^{\,j} m$ by m_j and $\partial_s^{\,j} l$ by l_j we also have

$$(\partial_t + \partial_s)\, m_j = l_j . \tag{9.7}$$

Notice that the support of l_j lies in the interval $-\rho < s < \rho$. Referring back to (7.11) and (7.13) we see that our task is to estimate $m_{(n-1)/2}$ for $s < \rho$ when n is odd and $J_+ m$ for $s < \rho$ when n is even. For this purpose we require L_2-estimates over the somewhat extended domain $(-\infty, 2\rho)$

$$I(\tau + 3\rho, j) = \int_{-\infty}^{2\rho} | m_j(s, \tau + 3\rho)|^2 \, ds = \int_{-\infty}^{-\rho} + \int_{-\rho}^{2\rho} \equiv I_1 + I_2 ;$$

we omit the ω variable and the ω integration is assumed but not indicated.

It is clear from the differential equation (9.7) and the fact that the support of l_j is in the interval $[-\rho, \rho]$ that for $s < -\rho$, m_j is constant along characteristics; that is,

$$m_j(s, \tau + 3\rho) = m_j(s - \tau - 3\rho, 0).$$

Hence,

$$I_1 = \int_{-\infty}^{-\tau-4\rho} | m_j(s, 0)|^2 \, ds.$$

Now $m_j(s, 0)$ is square integrable for $(n - 1)/2 \leqslant j \leqslant (n - 1)/2 + N - 1$. This is clear for n odd from (7.11) and for n even from (7.13) by making use of the trivial inequality

$$\| m \|_\nu^2 \leqslant \tfrac{1}{2} \| m \|_{\nu-1/2}^2 + \tfrac{1}{2} \| m \|_{\nu+1/2}^2 , \tag{9.8}$$

which holds for all real ν. It follows for such j that $I_1 < \epsilon$ for τ sufficiently large.

To estimate I_2 we integrate (9.7) along characteristics between $(-\rho, \tau + 2\rho - s)$ and $(s, \tau + 3\rho)$. For each s in the interval $[-\rho, 2\rho]$

$$m_j(s, \tau + 3\rho) = m_j(-\rho, \tau + 2\rho - s) + \int_0^{\rho+s} l_j(-\rho + t, \tau + 2\rho - s + t) \, dt,$$

where as above

$$m_j(-\rho, \tau + 2\rho - s) = m_j(-3\rho + s - \tau, 0).$$

Note that the integral involves only values of l_j for $\tau \leqslant t \leqslant \tau + 3\rho$. Applying (9.5) it is easy to see (using (9.8) for n even) that $I_2 = 0(\epsilon)$ for τ sufficiently large and $(n - 1)/2 \leqslant j \leqslant (n - 1)/2 + N - 1$.

When n is odd we take $N = 1$ and $j = (n - 1)/2$ and combining the above estimates for

$$\mathscr{J} P_+{}^\rho \xi T(t) \, Ag = \begin{cases} m_{(n-1)/2} & \text{for} \quad s < \rho, \\ 0 & \text{for} \quad s > \rho, \end{cases}$$

we see that $\| P_+{}^\rho T(\tau + 3\rho) \, Ag \|_E = 0(\epsilon^{1/2})$ for arbitrary large τ, as desired.

When n is even, J_+ is not a local operator, rather it is a half-order derivative which can be represented as a *convolution operator with support in* $\mathbb{R}_-$. In order to estimate $J_+ m$ we again avail ourselves of the trivial inequality (9.8). Our strategy will be to sandwich an estimate for $J_+ \partial_s m$ between estimates for $m_{n/2}$ and $m_{n/2+1}$, and these in turn will require estimates on the $D_+{}^\rho$ parts of $J_+ m$, $J_+ \partial_s m$ and $J_+ \partial_s{}^2 m$. We will take $N = 3$ in this case.

Our first task will be to "prepare" g so that the $D_+{}^\rho$ parts of $J_+ m$, $J_+ \partial_s m$, and $J_+ \partial_s{}^2 m$ are small. We start with an arbitrary f in $D(A^N)$.

Given $\epsilon > 0$ we choose τ_ϵ so that for $t > \tau_\epsilon$

$$\| P_+^\rho T(t) A^i f \|_E - \lim \| P_+^\rho T(t) A^i f \|_E < \epsilon \qquad \text{for} \quad i = 0, 1, 2, 3. \tag{9.9}$$

The data $A^i \xi T(t) f$ lie in H_0 as before and have outgoing translation representers $k_i = \partial_s{}^i k$, $i = 0, 1, 2, 3$ where $k = J_+ R\xi T(t) f$. Each of these functions satisfies an equation of the form

$$(\partial_t + \partial_s) k_i = \partial_s^{i-1} p, \tag{9.10}$$

where $p = J_+ R\{0, q\}$. Since q has its support in the ball $\{| x | < \rho/2\}$ it is easy to see from the form of J_+ that p will have its support on the half-line $(-\infty, \rho/2)$. Integrating (9.10) we see that

$$k_i(s, t) = k_i(s + \tau, t + \tau) \qquad \text{for} \quad s > \rho/2 \quad \text{and} \quad \tau > 0.$$

In other words once the data gets into $D_+^{\rho/2}$ it is simply translated to the right. As a matter of fact, the use of ξ above is merely a convenient way of obtaining an outgoing translation representation for $(1 - P_+^{\rho/2}) T(t) A^i f$ since $A^i \xi T(t) f = T(t) A^i f$ for $| x | > \rho/2$. Keeping this in mind we construct $\alpha \in C_0^\infty(R)$ such that

$$\alpha(s) = \begin{cases} 1 & \text{for} \quad 3\rho/4 < s < 5\rho/4, \\ 0 & \text{for} \quad s < \rho/2 \quad \text{and} \quad s > 3\rho/2. \end{cases}$$

We can then find arbitrary large t such that

$$\| \alpha k \| + \| \alpha k_1 \| + \| \alpha k_2 \| + \| \alpha k_3 \| < \epsilon, \tag{9.11}$$

since otherwise at least one of the three terms will be greater or equal to $\epsilon/4$ for an infinite subsequence of t's of the form $t = j\rho$ and, hence, will have infinite energy by the above. Note that αk_i lies in $D_+^{\rho/2}$ and, hence, in H.

We now choose $t_0 > \tau_\epsilon$ and such that (9.11) holds and denote by d the data in H corresponding to $\alpha k(t_0)$. Since $\mathscr{I}_+ A^i d = (-\partial_s)^i (\alpha k)$ it follows from (9.11) that

$$\| A^i d \|_E < \text{const}\, \epsilon \qquad \text{for} \quad i = 0, 1, 2, 3. \tag{9.12}$$

Further,

$$h = T(t_0) f - d \tag{9.13}$$

has the property that its outgoing translation representer vanishes near $s = \rho$. Since the action of P_+^ρ is to chop off at $s = \rho$ in this representation, it is clear that

$$P_+^\rho h \in D(A^3) \quad \text{and that} \quad A^i P_+^\rho h = P_+^\rho A^i h \qquad \text{for} \quad i = 1, 2, 3. \tag{9.14}$$

Finally, we set $g = P_+^\rho h$. It then follows from (9.12)–(9.14) that

$$\| A^i g - P_+^\rho T(t_0) A^i f \|_E < \text{const } \epsilon \qquad \text{for} \quad i = 0, 1, 2, 3. \tag{9.15}$$

Combining this with (4.2) we see for $s \geqslant 0$ that

$$\| P_+^\rho T(s) A^i g - P_+^\rho T(s + t_0) A^i f \|_E < \text{const } \epsilon$$

and, hence, that

$$| \lim \| P_+^\rho T(s) A^i g \|_E - \lim \| P_+^\rho T(s) A^i f \|_E | \leqslant \text{const } \epsilon. \tag{9.16}$$

Thus, in order to prove that $\lim \| P_+^\rho T(t) A^2 f \|_E = 0$ it suffices to prove that $\liminf \| P_+^\rho T(t) A^2 g \|_E$ is arbitrary small.

We also have at hand an estimate for $\|(I - P_+^\rho) T(t) A^i g \|_E$, which was the purpose of this elaborate preparation lemma. In fact,

$$\|(I - P_+^\rho) T(t) A^i g \|_E^2 = \| T(t) A^i g \|_E^2 - \| P_+^\rho T(t) A^i g \|_E^2$$

and using the estimates (9.9), (9.15), and (9.16) to obtain

$$\begin{aligned} \| T(t) A^i g \|_E &\leqslant \| A^i g \|_E \leqslant \| P_+^\rho T(t_0) A^i f \|_E + \text{const } \epsilon \\ &\leqslant \lim \| P_+^\rho T(t) A^i f \|_E + \text{const } \epsilon, \\ \| P_+^\rho T(t) A^i g \|_E &\geqslant \lim \| P_+^\rho T(t) A^i f \|_E - \text{const } \epsilon, \end{aligned}$$

we see that

$$\|(I - P_+^\rho) T(t) A^i g \|_E \leqslant \text{const } \epsilon^{1/2} \qquad \text{for} \quad i = 0, 1, 2, 3. \tag{9.17}$$

We now repeat the first part of this proof again with $m = R\xi T(t) Ag$. Our previous analysis provides us with an estimate for $\partial_s{}^j m(s, \tau + 3\rho)$ for $s < 2\rho$ and $j = n/2$ and $n/2 + 1$. We shall use (9.17) to estimate $\partial_s{}^j m(s, \tau + 3\rho)$ for all s. To this end we choose $\beta \in C^\infty(R)$ so that $\beta = 1$ for $s > 2\rho$ and $= 0$ for $s < \rho$. Then

$$\int | \beta J_+ \partial_s{}^i m |^2 \, ds \leqslant \|(I - P_+^\rho) T(t) A^{i+1} g \|_E^2 \leqslant \text{const } \epsilon^{1/2} \qquad \text{for} \quad i = 0, 1, 2.$$

Setting $e(t)$ equal to the data in H with outgoing translation representer $\beta J_+ m$, we see that

$$\| A^i e \|_E \leqslant \text{const } \epsilon^{1/2} \qquad \text{for} \quad i = 0, 1, 2. \tag{9.18}$$

Moreover, $\xi T(t) A^i g - A^i e(t)$ will be orthogonal to $D_+^{2\rho}$. Hence, if we set $m' = Re$ and $m'' = m - m'$, then it follows from Lemma 2.3

[7] that $\partial_s{}^j m''$ vanishes for $s > 2\rho$ if $j \geqslant n/2$. It follows from (9.8) and (9.18) that

$$\int_{-\infty}^{\infty} | \partial_s{}^j m' |^2 \, ds \leqslant \text{const } \epsilon \tag{9.19}$$

for $j = n/2$ and $n/2 + 1$. Further since m'' vanishes for $s > 2\rho$ we see from our previous analysis and (9.19) that

$$\int_{-\infty}^{\infty} | \partial_s{}^j m'' |^2 \, ds \leqslant 2 \int_{-\infty}^{2\rho} | \partial_s{}^j m |^2 \, ds + 2 \int_{-\infty}^{2\rho} | \partial_s{}^j m' |^2 \, ds \leqslant \text{const } \epsilon \tag{9.20}$$

for $t = \tau + 3\rho$ and $j = n/2$ and $n/2 + 1$. The inequalities (9.19) and (9.20) combine to give

$$\int_{-\infty}^{\infty} | \partial_s{}^j m |^2 \, ds \leqslant \text{const } \epsilon$$

for $t = \tau + 3\rho$ and $j = n/2$ and $n/2 + 1$. Applying (9.8) once again gives

$$\int | \partial_s J_+ m |^2 \, ds \leqslant \text{const } \epsilon$$

for $t = \tau + 3\rho$. In other words $\| A\xi T(\tau + 3\rho) Ag \|_E \leqslant \text{const } \epsilon^{1/2}$.

Finally,

$$T(\tau + 3\rho) A^2 g = A\xi T(\tau + 3\rho) Ag + A(1 - \xi) T(\tau + 3\rho) Ag.$$

The first term on the right has been estimated above and an estimate for the second term can be obtained directly from (9.5). This concludes the proof of Theorem 9.5 for n even.

10. The Scattering Matrix, n odd

We have now completed the verification for the acoustic equation of all the assumptions needed in the abstract theory of Part I. In this section we show how to construct the scattering matrix in analytical terms; we shall rely on the abstract theory developed at the end of Section 3 where an expression was found for the scattering matrix in terms of the function representing the outgoing component of the so-called reflected wave v.

We recall that for $\operatorname{Im} z < 0$, $\mathscr{S}(z)n$ is determined by the eigenfunctions e_0 and $e = W_1 e_0$ of A_0 and A, respectively, where the translation representations for e_0 and $e_\pm$ are given by

$$\mathscr{I} e_0 = e^{-izs} n, \qquad n \text{ in } N = L_2(S^{n-1}), \tag{10.1}$$

$$\mathscr{I} e_- = e^{-izs} n \text{ for } s < -\rho \qquad \text{and} \qquad \mathscr{I} e_+ = e^{-izs} \mathscr{S}(z) n \text{ for } s > \rho. \tag{10.2}$$

The reflected wave v was defined by (3.16) as

$$v = e - J_0 e_0 , \tag{10.3}$$

where J_0 can be realized as multiplication by a smooth function $j_0 \equiv 1$ for $|x| > \rho$ and vanishing near ∂G.

As a first step we determine, using formula (7.17), the eigenfunction $e_0 = \{u, u_t\}$ of A_0

$$u(x) = c_1(-iz)^{(n-3)/2} \int e^{-izx\cdot\omega} n(\omega)\, d\omega, \qquad c_1 = 2^{-1/2}(-1/2\pi)^{(n-1)/2}, \tag{10.4}$$

$$u_t = izu. \tag{10.5}$$

As shown in (3.17), v satisfies the equation

$$(iz - A)v = (A - iz)\, J_0 e_0 ; \tag{10.6}$$

and for $\operatorname{Im} z < 0$, v is the unique solution in H of (10.6).

Away from the boundary the operator A is defined by

$$A = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix},$$

and a brief calculation allows us to deduce from (10.6) the following equations for the components of $v = \{w, w_t\}$:

$$w_t = izw, \tag{10.7}$$

$$Lw = -Lj_0 u, \tag{10.8}$$

where u is the first component of e_0 and L stands for the operator

$$L = \Delta + z^2. \tag{10.9}$$

Data in the domain of A also satisfy the boundary condition (8.1)

$$w_n + \alpha w_t = 0,$$

which because of (10.7) can be rewritten as

$$w_n + i\alpha z w = 0 \qquad \text{on} \quad \partial G. \tag{10.10}$$

As remarked in Section 3 for $\operatorname{Im} z < 0$, v and Av belong to H; this means that $\partial_x^{\beta} w$ and $\partial_t w = izw$ are square integrable in G for $|\beta| = 1, 2$. We now extend w by extrapolation into the inside of the obstacle so as to preserve these properties and rewrite (10.8) as

$$Lw = g \qquad \text{over} \quad \mathbb{R}^n, \tag{10.11}$$

where g abbreviates $-Lj_0 u$. The function u defined by (10.4) satisfies

$$Lu = 0, \tag{10.12}$$

and it follows that *g vanishes for $|x| > \rho$.*

Our immediate goal is to determine the translation representation for v_+, the outgoing part of v, in terms of the asymptotic behavior of w for large $|x|$. More precisely we shall prove the following theorem.

LEMMA 10.1. *For large $|x|$ the function w has the asymptotic behavior*

$$w(x) = \frac{e^{-iz|x|}}{|x|^{(n-1)/2}}\left[k(\mathcal{O}) + O\left(\frac{1}{|x|}\right)\right] \tag{10.13}$$

where $\mathcal{O} = x/|x|$ and

$$\mathscr{I}v_+ = (-1)^{(n-3)/2}\sqrt{2}\, ize^{-izs}k(\mathcal{O}) \qquad \text{for} \quad s > \rho. \tag{10.14}$$

As explained at the end of Section 3, we can deduce from (10.3) that

$$\mathscr{I}v_+ = e^{-izs}(\mathscr{S}(z)n - n) \qquad \text{for} \quad s > \rho,$$

and, hence, it follows from (10.13) and (10.14) that

$$\mathscr{S}(z)n = n + \sqrt{2}\,(-1)^{(n-3)/2}\, izk(\mathcal{O}). \tag{10.15}$$

Before starting on this program it is useful to take another look at (10.4). This formula expresses u as a linear combination of eigenfunctions $e_0' = \{u', u_t'\}$ of A_0

$$u'(x, \omega, z) = e^{-izx\cdot\omega}, \tag{10.16}$$

$$u_t' = izu'. \tag{10.17}$$

Suppose we now define v' and w' from u' exactly as v and w were defined above from u. Then w' is the square integrable solution of the boundary value problem corresponding to (10.8), and we extend w' into the interior of the obstacle in a fashion which is uniform in ω. It is clear that the w' will be uniformly square integrable with respect to the parameter ω. Since

$$u(x) = c_1(-iz)^{(n-3)/2}\int u'(x, \omega)\, n(\omega)\, d\omega,$$

we see that v is the superposition with weight $n(\omega)$ of the uniformly square integrable $v'(\cdot, \omega)$

$$v(x) = c_1(-iz)^{(n-3)/2}\int v'(x, \omega)\, n(\omega)\, d\omega$$

and that

$$w(x) = c_1(-iz)^{(n-3)/2}\int w'(x, \omega)\, n(\omega)\, d\omega. \tag{10.18}$$

If the asymptotic behavior of w' is given by

$$w'(x, \omega, z) = \frac{e^{-izs}}{|x|^{(n-1)/2}} \left[k'(\mathcal{O}, \omega, z) + O\left(\frac{1}{|x|}\right)\right] \tag{10.19}$$

then we see by (10.13), (10.18), and (10.19) that

$$k(\mathcal{O}) = c_1(-iz)^{(n-3)/2} \int k'(\mathcal{O}, \omega, z)\, n(\omega)\, d\omega. \tag{10.20}$$

Combining (10.20) with (10.15) gives the following lemma.

LEMMA 10.2. *For* $\operatorname{Im} z < 0$

$$\mathcal{S}(z)n = n + (-iz/2\pi)^{(n-1)/2} \int k'(\mathcal{O}, \omega, z)\, n(\omega)\, d\omega. \tag{10.21}$$

In the proof of Lemma 10.1 we need the following integral representation for w (we omit the primes from now on).

LEMMA 10.3. *The function w has the representation*

$$w(x) = \int_{|y|<\rho} g(y)\, \gamma(x-y)\, dy \tag{10.22}$$

for $|x| > \rho$; where γ is the fundamental solution for L. Expressed in terms of Hankel functions γ is

$$\gamma(x, z) = (i/4)(z/2\pi\, |x|)^{(n-2)/2}\, H^{(2)}_{(n-2)/2}(z\, |x|). \tag{10.23}$$

Remark. In particular when $n = 3$,

$$\gamma(x) = -e^{-iz|x|}/(4\pi\, |x|).$$

Proof. The function $\gamma(x)$ satisfies the relation

$$L\gamma = \delta(x). \tag{10.24}$$

We apply Green's formula to the functions $\gamma(x-y)$ and $w(y)$ in the ball $B = \{|y| < R\}$ and obtain

$$\int_B [\gamma(x-y)\, Lw - wL\gamma(x-y)]\, dy = \int_{\partial B} [\gamma(x-y)\, w_n - \gamma_n(x-y) w]\, dS.$$

It is seen from (10.23) for $\operatorname{Im} z < 0$ that $\gamma(x-y)$ tend to zero exponentially as $|x|$ tends to infinity. Since w and $\partial_x w$ are square integrable, it follows that the boundary terms in (10.25) tend to zero as R tends to infinity. Using the fact that $Lw = g$, here $g(x) = 0$ for $|x| > \rho$, and the relation (10.24), we obtain the formula (10.22) for $|x| > \rho$. This completes the proof of Lemma 10.3.

Proof of Lemma 10.1. We first show that w has the asymptotic form (10.13). Using the relation

$$|x - y| = |x| - y \cdot x/|x| + O(1/|x|),$$

we deduce from the asymptotic behavior of the Hankle function that

$$\gamma(x - y) = -\frac{1}{4\pi}\left(\frac{iz}{2\pi}\right)^{(n-3)/2} \frac{e^{-iz|x|}}{|x|^{(n-1)/2}}\left\{e^{izy\cdot\mathcal{O}} + O\left(\frac{1}{|x|}\right)\right\}.$$

Substituting this into (10.22) we get

$$w(x) = -\frac{1}{4\pi}\left(\frac{iz}{2\pi}\right)^{(n-3)/2} \frac{e^{-iz|x|}}{|x|^{(n-1)/2}}\left\{\int e^{izy\cdot\mathcal{O}} g(y)\, dy + O\left(\frac{1}{|x|}\right)\right\},$$

which is (10.13) with

$$k(\mathcal{O}) = -\frac{1}{4\pi}\left(\frac{iz}{2\pi}\right)^{(n-3)/2} \int e^{izy\cdot\mathcal{O}} g(y)\, dy. \tag{10.26}$$

Next we determine the translation representation of v_+. As we explained at the end of Section 8, only the values of v outside the ball $\{|x| < \rho\}$ effect the translation representation of v_+. Thus, it does not matter whether we use $\{w, izw\}$ for the given w or the extended w, v_+ is uneffected. If we use the extended w then the corresponding v is in the domain of the unperturbed generator A_0 and for this extended v we have

$$(iz - A_0)v = \{0, -g\}. \tag{10.27}$$

We can now use the formulas of Section 7. By (7.11)

$$\mathscr{I}v = (1/\sqrt{2})\, \partial_s^{(n-1)/2} Rv \tag{10.28}$$

and combining this with (7.9) we can rewrite (10.27) as

$$(iz + \partial_s)\, \mathscr{I}v = \mathscr{I}\{0, -g\} \equiv l. \tag{10.29}$$

Since g has its support in the ball $\{|x| < \rho\}$, the translation representer l will have its support on the interval $-\rho < s < \rho$. Thus, on integrating (10.29) we see for the extended v that

$$\mathscr{I}v = e^{-izs}m_- \quad \text{for} \quad s < -\rho, \ = e^{-isz}m_+ \quad \text{for} \quad s > \rho.$$

However, since $\operatorname{Im} z < 0$ and since the extended v has finite energy we conclude that $m_- = 0$ (this checks with v being outgoing) and, hence, that

$$\mathscr{I}v_+(s, \mathcal{O}) = e^{-izs} \int_{-\rho}^{0} e^{izt} l(t, \mathcal{O})\, dt \qquad \text{for} \quad s > \rho. \tag{10.30}$$

Using the fact that l is the translation representer of $\{0, g\}$, the relation (10.30) can be evaluated directly in terms of g by means of (7.4), (7.8) and (7.11) as

$$\mathscr{I}v_+(s, \mathcal{O}) = \frac{e^{-izs}}{\sqrt{2}\,(2\pi)^{(n-1)/2}} \int_{-\rho}^{\rho} e^{izt} \partial_t^{(n-1)/2} \left[\int\!\!\int_{y\cdot\mathcal{O}=t} g(y)\, dS\right] dt. \tag{10.31}$$

Using the fact that g has support in the ball $\{|\, y\,| < \rho\}$, we can integrate (10.31) by parts to obtain

$$\begin{aligned}\mathscr{I}v_+(s, \mathcal{O}) &= 1/\sqrt{2}\,(-iz/2\pi)^{(n-1)/2}\, e^{-izs} \int_{-\rho}^{\rho} e^{izt} \left[\int\!\!\int_{y\cdot\mathcal{O}=t} g(y)\, dS\right] dt \\ &= 1/\sqrt{2}\,(-iz/2\pi)^{(n-1)/2}\, e^{-izs} \int e^{izy\cdot\mathcal{O}} g(y)\, dy.\end{aligned} \tag{10.32}$$

Combining this with (10.26) we finally obtain (10.14). This completes the proof of Lemma 10.1.

So far we have obtained the representations (10.21) of the scattering matrix only for $\operatorname{Im} z < 0$. It is of the form

$$\mathscr{S}(z) = I + (-iz/2\pi)^{(n-1)/2}\, \mathscr{K}', \tag{10.33}$$

where $\mathscr{K}'$ is an integral operator whose kernel $k'(\mathcal{O}, w, z)$ is determined via formula (10.22) by the asymptotic description for large $|\, x\,|$ of the L^2 solution w' of the problem (10.11); here g' depends in the first place on u' of (10.16), then on the solution v' of (10.6) and finally on the extension of w' inside the obstacle. We now proceed to extend this result into the upper half plane.

Replacing e_0 in (10.6) by e_0' of (10.16) and (10.17) we see that v' is the outgoing solution of

$$(iz - A)\, v' = \{0, Lj_0 u_0'\}, \tag{10.34}$$

where Lj_0u_0' is smooth analytic in z and with support in $|\, x\,| < \rho$. We now appeal to Theorem 4.4 according to which *there exists a unique outgoing solution of* (10.34) *in* $\bar{H}$. Further, it is obvious from the proof of this theorem that $v'(z)$ is locally holomorphic on $-i\rho(B)$

since it depends only on $(iz - B)^{-1}\{0, Lj_0u_0'\}$. Setting $v' = \{w', w_t'\}$ we then extend w' by extrapolation into the obstacle. This too can be done in a way which preserves the analytic dependence on z. We conclude that the extended w' satisfies

$$Lw' = g' \qquad \text{over} \quad \mathbb{R}^n \tag{10.11$'$}$$

and that $g'(z)$ *is holomorphic on* $-i\rho(B)$ *to* $L_2(\mathbb{R}^n)$ *with support in* $|x| < \rho$. We note that g' also depends continuously on ω.

LEMMA 10.4. *For all* z *in* $-i\rho(B)$, *the function* $w'(z)$ *has the representation*

$$w'(x, z) = \int_{|y|<\rho} g'(y, z)\,\gamma(x - y, z)\,dy \tag{10.22$'$}$$

for $|x| > \rho$.

Proof. It follows from Lemma 10.3 that the assertion holds for all z in the lower half plane. As we have just shown both sides of the equation are analytic in $-i\rho(B)$, and, consequently, the relation continues to hold in the extended domain.

As in the proof of Lemma 10.1 we can readily obtain the asymptotic behaviors of w' from (10.22)$'$.

LEMMA 10.5. *For large* $|x|$ *the function* w' *has the asymptotic behavior*

$$w'(x, \omega, z) = \frac{e^{-iz|x|}}{|x|^{(n-1)/2}} \left\{k'(\mathcal{O}, \omega, z) + O\left(\frac{1}{|x|}\right)\right\}, \tag{10.13$'$}$$

where

$$k'(\mathcal{O}, \omega, z) = -\frac{1}{4\pi}\left(\frac{iz}{2\pi}\right)^{(n-3)/2} \int e^{izy\cdot\mathcal{O}} g'(y, \omega, z)\,dz. \tag{10.26$'$}$$

It is clear from (10.26)$'$ that $k'(\mathcal{O}, z)$ is also analytic in $-i\rho(B)$. If we now include the dependence on ω, then we see from Lemma 10.2 and the fact that $\mathcal{S}(z)$ is also holomorphic in $-i\rho(R)$ that (10.26)$'$ gives us a concrete way of determining the integral operator $\mathcal{K}'$ that occurs in the expression (10.21) for $\mathcal{S}(z)$.

According to Lemma 5.2 the solution to (10.34) furnished us by Theorem 4.11 is unique within the class of $\bar{H}$ data and can be represented as a superposition $\gamma(x - y)$'s. We would like now to prove that it is unique within the class of solutions representable in the form (10.22)$'$. For this to be so it suffices to show that any such solution has a translation representation in $\bar{H}$. Now $v - J_0 v$ lies in $K^a \subset \bar{H}$. Hence, by considering $J_0 v$ instead of v we can reduce the problem to the following lemma.

LEMMA 10.6. *Suppose g is in L_2 with support in $\{|x| < \rho\}$ and set*

$$w(x, z) = \int_{|x|<\rho} g(y)\, \gamma(x - y, z)\, dy. \tag{10.22}$$

Then $v = \{w, izw\}$ has a translation representation in $\bar{H}_0$.

Proof. It is easy to construct an outgoing solution to

$$(iz - A_0)v = \{0, g\} \tag{10.35}$$

in $\bar{H}_0$. In fact if we consider the $\bar{H}_0$ representation of (10.35) we get

$$(iz + \partial_s)\, \mathscr{I}v = l = \mathscr{I}\{0, g\}, \tag{10.36}$$

where the support of l lies in $-\rho < s < \rho$. The unique outgoing solution of (10.36) is simply

$$\mathscr{I}v(s) = \int_{-\rho}^{s} e^{-iz(s-t)} l(t)\, dt \qquad \text{for} \quad s > -\rho. \tag{10.37}$$

It is clear that $\mathscr{I}v$ and v (obtained from $\mathscr{I}v$ by (7.14)) are holomorphic in the respective local topologies of $\bar{H}_0$ and H_0.

Because of the fact that l has support in $-\rho < s < \rho$, it is evident from (10.37) that for $\operatorname{Im} z < 0$, v has finite energy and it is known that the only solution of (10.35) in H_0 is given by (10.22). Thus, for $\operatorname{Im} z < 0$, the data (w, izw), given by (10.22), has the translation representation (10.37). Since both (w, izw), given by (10.22), and v definedly (10.37) are holomorphic in the entire plane, they must coincide for all z. This completes the proof of Lemma 10.6.

Before stating the main theorem of this section we make one final reduction. Instead of considering w' it is customary to work with

$$q' = w' + (j_0 - 1)\, u'. \tag{10.38}$$

Since w' satisfies Eq. (10.8)

$$Lw' = -Lj_0 u' \qquad \text{in} \quad G$$

with boundary condition (10.10)

$$w_n' + iz\alpha w' = 0 \qquad \text{in} \quad \partial G,$$

we see from (10.16) that

$$Lq' = 0 \quad \text{in} \quad G \tag{10.39}$$

and

$$q_n' + iz\alpha q' = u_n' + iz\alpha u'. \tag{10.40}$$

Since $j_0 \equiv 1$ for $|x| > \rho$, $q' = w'$ for $|x| > \rho$, and, hence, q' also has an integral representation of the kind (10.22)'.

We summarize all of the above results in the following theorem.

THEOREM 10.7. *For all z in the domain of regularity of $\mathscr{S}$; the exterior boundary value problem* (10.39) *and* (10.40) *has a unique outgoing solution q', that is one which has an integral representation of the form* (10.22). *For all such z the scattering matrix $\mathscr{S}(z)$ is of the form* (10.21) *where k' is determined by the asymptotic behavior near ∞ of the outgoing solution q'.*

Remark. The above analysis shows more generally that the boundary value problem

$$Lu = 0 \text{ in } G, \quad u_n + iz\alpha u \text{ prescribed on } \partial G$$

has a unique outgoing solution for z in the domain of regularity of $\mathscr{S}$.

The adjoint system is in essence the same as the given system and can be treated in a similar fashion. If we denote the corresponding wave operators by

$$W_1{}^0 = \text{st lim } T^*(t)\, J_0 U_0(t),$$

$$W_2{}^0 = \text{st lim } U_0(t)\, JT^*(t),$$

then it is clear that

$$W_1{}^* = W_2{}^0 \qquad \text{and} \qquad W_2{}^* = W_1{}^0. \tag{10.41}$$

The scattering operator for the adjoint system is

$$S^0 = W_2{}^0 W_1{}^0,$$

and combining this with (1.15) and (10.41) we get

$$S^0 = S^*. \tag{10.42}$$

It follows that the corresponding scattering matrices are related for real σ by

$$\mathscr{S}^0(\sigma) = \mathscr{S}^*(\sigma),$$

and continuing this relation analytically we obtain

$$\mathscr{S}^0(z) = \mathscr{S}^*(\bar{z}). \tag{10.43}$$

Comparing the kernels of the integral operators on both sides of (10.43) gives us the reciprocity law

$$k^0(\theta, \omega, z) = \overline{k'(\omega, \theta, \bar{z})}. \tag{10.44}$$

It is clear from (10.33) that $\mathscr{S}(z)$ is of the form $I + \mathscr{K}'(z)$ where $\mathscr{K}'$ is an integral operator. We see from (10.26)′ that the kernel of this operator is continuous on S^{n-1} and it follows that $\mathscr{K}'$ is a compact operator. Consequently, $\mathscr{S}(z)$ will have a bounded inverse if it is merely one-to-one. At $z = 0$ things are particularly simple and we have the following lemma.

LEMMA 10.8. *$\mathscr{S}(0) = I$ and $\mathscr{S}'(0) = icP$ where c is a real number satisfying $0 \leqslant c \leqslant 2\rho$ and P is the orthogonal projection of $L_2(S^{n-1})$ onto the constant functions; here ρ is chosen so that the obstacle is contained in the ball $\{|x| < \rho\}$.*

Proof. What is required is the transmission coefficient obtainable from the outgoing solution of the equation

$$\Delta v = 0 \text{ in } G, \qquad \partial_n v = 0 \text{ on } \partial G.$$

For $n = 3$ it is shown in Lemma 2.3 [8] for large $r = |x|$ that

$$v(r\theta) \sim k_0/r,$$

where $0 \geqslant k_0 \geqslant -\rho$. Because of the factor of $z^{(n-1)/2}$ it is clear that $\mathscr{S}(0) = I$ for $n > 1$ and $\mathscr{S}'(0) = 0$ for $n > 3$. For $n = 3$ we see that

$$\mathscr{S}'(0)n = -(i/2\pi)\, k_0 \int n(\omega)\, d\omega = icPn.$$

THEOREM 10.9. *$\mathscr{S}^{-1}(z)$ is meromorphic on $-i\rho(B)$.*

Proof. This follows from the fact that $\mathscr{K}'(z)$ is a compact-valued holomorphic function on $-i\rho(B)$ together with $\mathscr{S}(z)$ being invertible at $z = 0$.

THEOREM 10.10. *$\mathscr{S}(z)$ is invertible for all z in the circle of radius $1/2\rho$ about the point $(0, -i/2\rho)$.*

Proof. It follows from Theorem 3.1, Corollary 4.8, Theorem 5.5, Theorem 8.5, and Lemma 10.8 that

$$\varphi(z) \equiv e^{-i2\rho z}\mathcal{S}(z)$$

is holomorphic on the closed lower half plane $\operatorname{Im} z \leqslant 0$ and satisfies the properties: (i) $|\varphi(z)| \leqslant 1$ for $\operatorname{Im} z \leqslant 0$, (2) $\varphi(0) = I$, and (3) $\varphi'(0) = icP - 2i\rho I$ which is of norm 2ρ. The proof of Corollary 2.2 of [8] now suffices to establish the assertion of this theorem.

We close this section with a verification of the assumptions of Theorem 6.2. It will then follow from Theorems 6.1, 6.2, and 10.9, and the following lemma that

THEOREM 10.11. *The spectrum of A consists of a continuous part which fills out the imaginary axis plus a discrete point spectrum in the left half plane* $\operatorname{Re} \lambda < 0$.

LEMMA 10.12. *Let A be defined as in Section* 8 *and let A_1 be a special instance of A with Neumann boundary conditions on ∂G. Then for $\lambda > 0$, $R_\lambda(A) - R_\lambda(A_1)$ is a compact operator on H.*

Proof. It is clear that A_1 is skew self-adjoint. It will suffice to show that there exist compact linear operators Q_1 and Q_2 such that for $\operatorname{Re} \lambda > 0$

$$R_\lambda(A) = R_\lambda(A_1)(I - Q_1) - Q_2\,. \tag{10.45}$$

Our plan is to start with $v = R_\lambda(A_1)g$ and perturb v with a function pair of bounded support so that the resulting data satisfies the boundary condition (8.1) of A. Choose $\xi \in C^\infty(R_E)$ so that $\xi \equiv 1$ near ∂G and $\xi \equiv 0$ for $|x| > \rho - \delta$ for $\delta > 0$ sufficiently small. The perturbing function pair is then ξh, where

$$\begin{aligned} \left(\lambda - \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}\right) h &= 0 && \text{on } G_\rho, \\ \partial_n h_1 + \alpha h_2 &= \partial_n v_1 + \alpha v_2 && \text{on } \partial G, \\ h_1 &= 0 && \text{on } \{|x| = \rho\}; \end{aligned} \tag{10.46}$$

then h is indirectly a function of g. Note that $w = h - \xi v$ satisfies

$$\left(\lambda - \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}\right) w = \begin{pmatrix} -\xi g_1 \\ -\xi g_2 + (\Delta\xi)\, v_1 + 2\partial_x \xi \partial_x v_1 \end{pmatrix} \equiv g' \text{ in } G_\rho\,,$$

$$\partial_n w_1 + \alpha w_2 = 0 \qquad \text{on} \quad \partial G,$$

$$w_1 = 0 \qquad \text{on} \quad \{|x| = \rho\}.$$

it follows by elliptic theory that

$$\sum_{|\beta|\leqslant 2} \int |\partial_x^{\beta} w_1|^2 \, dx + \sum_{|\beta|\leqslant 1} \int |\partial_x^{\beta} w_2|^2 \, dx \leqslant c' \| g' \|_E^2 \leqslant c \| g \|_E^2$$

and combining this with Theorem 8.1 we obtain

$$\sum_{|\beta|\leqslant 2} \int |\partial_x^{\beta} h_1|^2 \, dx + \sum_{|\beta|\leqslant 1} \int |\partial_x^{\beta} h_2|^2 \, dx \leqslant c \| g \|_E^2 . \tag{10.47}$$

Next we set $u = v - \xi h$; then u lies in $D(A)$ and

$$\begin{aligned} f \equiv (\lambda - A)u &= g - \{0, -(\Delta \xi)\, h_1 - 2\partial_x \xi \partial_x h_1\} \\ &\equiv g - T_1 g; \end{aligned} \tag{10.48}$$

this defines the operator T_1. Since $T_1 g$ involves only h_1 and its first derivatives, it follows from (10.47) and the Rellich compactness criterion that T_1 is a compact operator. Similarly $T_2 g = \xi h$ is compact. Now

$$R_\lambda(A) f = u = v - \xi h = R_\lambda(A_1) g - T_2 g,$$

and since $f = (1 - T_1) g$ it is enough to prove that $I - T_1$ is one-to-one. For in this case $(I - T_1)^{-1}$ exists and is of the form $I - Q_1$ where Q_1 is compact. Further, $Q_2 = T_2(I - T_1)^{-1}$ will also be compact, and, hence, we will have established (10.45).

In order to prove that $I - T_1$ is one-to-one we suppose the contrary; that is, suppose $g \in H$ is such that $g = T_1 g$. According to (10.48) we will then have $(\lambda - A)u = 0$, and since A is dissipative we may conclude that $u = 0$. In this case v is identically equal to h near ∂G. This implies $\partial_n h_1 = \partial_n v_1 = 0$ on ∂G. Thus,

$$\left(\lambda - \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}\right) h = 0 \qquad \text{in} \quad G_\rho ,$$

$$\partial_n h_1 = 0 \text{ on } \partial G \qquad \text{and} \qquad h_1 = 0 \text{ on } \{|x| = \rho\}.$$

This is a familiar interior elliptic problem which is known to have only the trivial solution. Thus, $h = 0$, $T_1 g = 0$ and hence $g = 0$, as desired for uniqueness. This completes the proof of Lemma 10.12.

APPENDIX—An Intrinsic Scattering Theory

The material in this paper can also be presented in a way which is independent of the unperturbed group. In this case we start with a semigroup of contraction operators $T(t)$ on a Hilbert space H containing two distinguished closed subspaces D_+ and D_- such that

(i) $T(t)\, D_+ \subset D_+$ and $T^*(t)\, D_- \subset D_-$;

(ii) $\wedge\, T(t)\, D_+ = \{0\}$ and $\wedge\, T^*(t)\, D_- = \{0\}$;

(iii) $\operatorname{st\,lim} P_+ T(t) = 0$ and $\operatorname{st\,lim} P_- T^*(t) = 0$;

(iv) $T(t)$ is isometric on D_+ and $T^*(t)$ is isometric on D_-.

Following the material in Chapter 2 [6] we construct unitary translation representations for D_+ and D_-

$$\begin{aligned} &D_+ \to L_2(\mathbb{R}_+ + \rho, N_+),\\ &T(t) \to \text{right translation by } t \text{ units;} \end{aligned} \tag{B1}$$

$$\begin{aligned} &D_- \to L_2(\mathbb{R}_- - \rho, N_-),\\ &T^*(t) \to \text{left translation by } t \text{ units.} \end{aligned} \tag{B2}$$

We now define $U_+(t)$ on $H_+ = L_2(\mathbb{R}, N_+)$ and $U_-(t)$ on $H_- = L_2(\mathbb{R}, N_-)$ so that the action of both $U_+(t)$ and $U_-(t)$ is right translation by t units. Choose J_- to be the orthogonal projection of H_- onto D_- and J_+ that of H onto D_+.

Then as in Section 1 it is easy to prove that the following wave operators exist

$$W_1 = \operatorname{st\,lim} T(t)\, J_- U_-(-t), \tag{B3}$$

$$W_2 = \operatorname{st\,lim} U_+(-t)\, J_+ T(t). \tag{B4}$$

We define the scattering operator

$$S = W_2 W_1 \tag{B5}$$

on H_- to H_+.

Finally, if D_- is orthogonal to D_+ and N_- is unitarily equivalent with N_+, then we can *identify* H_- with H_+. Call the common space H_0 and the common unitary operator $U_0(t)$. The abstract theory of Part I then proceeds as before.

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Printed in Belgium